Math Competitions Corner

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No. 2

This section of the Journal offers readers an opportunity to solve interesting mathematical problems appeared previously in High School Mathematical Olympiads and University Competitions or used by trainers and contestants to prepare Math Competitions. Elegant solutions, generalizations of the problems posed and new suitable proposals are always welcomed. Proposals should be accompanied by solutions. The origin of the problems appeared previously will be revealed when the solutions are published.

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Solutions to the problem stated in this issue should be posted before May 15, 2003

PROBLEMS

MC–21. Find all real solutions of the following system of equations

\[
\begin{align*}
  x_1 x_2 + 1 &= 4x_2, \\
  x_2 x_3 + 1 &= x_3, \\
  x_3 x_4 + 1 &= 4x_4, \\
  &\vdots \\
  x_{2011} x_{2012} + 1 &= 4x_{2012}, \\
  x_{2012} x_1 + 1 &= x_1.
\end{align*}
\]

\[25]^2

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MC–22. Is there a polynomial $A(x)$ with real coefficients such that

$$A(m) = 2012^m$$

for all positive integer $m$?

MC–23. Find all the integers $a > 1$ such that any prime divisor of $a^6 - 1$ is a divisor either of $a^3 - 1$ or $a^2 - 1$.

MC–24. Suppose $N$ is the sum of the squares of three positive integers. For all positive integer $n$, show that $N^{2^n}$ is also the sum of the squares of three positive integers.

MC–25. Let $a, b, c$ be the lengths of the sides of a triangle $ABC$. Prove that

$$\sqrt{\frac{a + b - c}{a + b + c}} + \sqrt{\frac{b + c - a}{a + b + c}} + \sqrt{\frac{c + a - b}{a + b + c}} \leq \sqrt{3}$$

MC–26. Let $k$ be a positive integer. Compute

$$\frac{1}{k!} \sum_{n_1 = 1}^{\infty} \sum_{n_2 = 1}^{\infty} \ldots \sum_{n_k = 1}^{\infty} \frac{1}{n_1 n_2 \ldots n_k (n_1 + n_2 + \ldots + n_k + 1)}$$

MC–27. Let $P_0, P_1, P_2, \ldots$ be a sequence of convex polygons such that, for each $k \geq 0$, the vertices of $P_{k+1}$ are the midpoints of all sides of $P_k$. Prove that there exists a unique point lying inside all these polygons.

MC–28. Calculate

$$\lim_{n \to \infty} \frac{1}{2^n} \sum_{k=0}^{n} \left\{ (-1)^k \binom{n}{k} \frac{\sqrt{k}}{\sqrt{k + \sqrt{n}}} \right\}$$

MC–29. Let $a, b, c$ be three complex numbers such that $a|bc| + b|ca| + c|ab| = 0$. Prove that

$$|(a - b)(b - c)(c - a) \geq 3\sqrt{3}|abc|$$
MC–30. Let $n$ be a positive integer. Compute

$$S(n) =$$

$$= \sum_{k=1}^{n} \left[ \arctg \left( \frac{k^2 + k - 1}{(k^2 + k + 2)(k^2 + k + 1)} \right) \arctg \left( \frac{k^4 + 2k^3 + 2k^2 + k + 2}{(2k + 1)(k^2 + k + 1)} \right) \right]$$

MC–31. Determine whether the real number $\frac{\ln(11 + 5\sqrt{2})}{\ln(5 + 11\sqrt{2})}$ is rational or not.

MC–32. Prove that for all $A \in \mathcal{M}_2(\mathbb{R})$ there exist $X, Y \in \mathcal{M}_2(\mathbb{R})$ such that $A = X^3 + Y^3$ and $XY = YX$.

MC–33. Let $a_1, a_2, \ldots, a_n$, and $b_1, b_2, \ldots, b_n$ be positive real numbers. Prove that

$$\left( \sum_{i=1}^{n} b_i \right)^{\sum_{i=1}^{n} b_i} \prod_{i=1}^{n} (a_i + b_i)^{b_i} \leq \left( \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i \right)^{\sum_{i=1}^{n} b_i} \prod_{i=1}^{n} b_i^{b_i}$$

When does equality occur?

MC–34. Let $A_1, A_2, \ldots, A_n$ be finite sets. Prove that

$$\left| \bigcap_{i=1}^{n} A_i \right| = \sum_{i} |A_i| - \sum_{i<j} |A_i \cup A_j| + \sum_{i<j<k} |A_i \cup A_j \cup A_k| - \ldots + (-1)^{n+1} \left| \bigcup_{i=1}^{n} A_i \right|$$

MC–35. Compute

$$\lim_{n \to \infty} \prod_{i=1}^{n} \left( \sum_{k=1}^{i} \frac{3k^2 + 9k + 7}{(k + 1)^3(k + 2)^3} \right)$$
SOLUTIONS

No problem is ever permanently closed. We will be very pleased considering for publication new solutions or comments on the past problems.

MC–1. Find all functions \( f : \mathbb{N} \to [0, +\infty) \) such that \( f(1000) = 10 \) and

\[
 f(n + 1) = \sum_{k=0}^{n} \frac{1}{f^2(k) + f(k)f(n + 1) + f^2(n + 1)}
\]

for all integer \( n \geq 0 \). (Here, \( f^2(i) \) means \((f(i))^2\).)

(IMAC – 2011)

Solution 1. Note that

\[
 f(n + 1) = \sum_{k=0}^{n-1} \frac{1}{f^2(k) + f(k)f(n + 1) + f^2(n + 1)} + \frac{1}{f^2(n) + f(n)f(n + 1) + f^2(n + 1)}
 = f(n) + \frac{1}{f(n) + f(n + 1)}
\]

From where

\[
 (f(n + 1) - f(n)) \left(f^2(n) + f(n)f(n + 1) + f^2(n + 1)\right) = 1
\]

Implying that

\[
 f^3(n + 1) - f^3(n) = 1.
\]

That is, \( f^3(n) = f^3(n + 1) - 1 \). Since \( f(1000) = 10 = \sqrt[3]{1000} \), then by induction it is easy to obtain that the function \( f \) is \( f(n) = \sqrt[3]{n} \) for \( n \in \mathbb{N} \): if \( f(m) = \sqrt[3]{m} \), for \( m > 1 \), then it also holds that \( f(m - 1) = \sqrt[3]{f^3(m) - 1} = \sqrt[3]{m - 1} \) and \( f(m + 1) = \sqrt[3]{f^3(m) + 1} = \sqrt[3]{m + 1} \).

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Solution 2. We have

\[ f(n + 1) - f(n) = \sum_{k=0}^{n} \frac{1}{f^2(k) + f(k)f(k+1) + f^2(k+1)} - \sum_{k=0}^{n-1} \frac{1}{f^2(k) + f(k)f(k+1) + f^2(k+1)} = \frac{1}{f^2(n) + f(n)f(n+1) + f^2(n+1)} \]

Rearranging terms, we get \( f^3(n + 1) - f^3(n) = 1 \) from which follows

\[ f^3(n + 1) = 1 + f^3(n) = 2 + f^3(n - 1) = \ldots = (n + 1) + f^3(0) \]

Putting \( n = 999 \), we get \( f^3(1000) = 1000 + f^3(0) \) from which follows \( f^3(0) = 0 \) and \( f(0) = 0 \). So, \( f^3(n + 1) = n + 1 \) and \( f(n) = \sqrt[3]{n} \).

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Also solved by Mohammad W. Alomari, Jerash University, Jordan, Arnau Mesegueré, Barcelona Tech, Barcelona, Spain and José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain.

MC–2. Let \( ABDC \) be a cyclic quadrilateral inscribed in a circle \( C \). Let \( M \) and \( N \) be the midpoints of the arcs \( AB \) and \( CD \) which do not contain \( C \) and \( A \) respectively. If \( MN \) meets side \( AB \) at \( P \), then show that

\[ \frac{AP}{BP} = \frac{AC + AD}{BC + BD} \]

(IMAC – 2011)

Solution. Applying Ptolemy’s theorem to the inscribed quadrilateral \( ACND \), we have

\[ AD \cdot CN + AC \cdot ND = AN \cdot CD \]
Since $N$ is the midpoint of the arc $CD$, then we have $CN = ND = x$, and $AN \cdot CD = (AC + AD)x$. Likewise, considering the inscribed quadrilateral $CNDB$ we have $BN \cdot CD = (BD + BC)x$. Dividing the preceding expressions, yields

$$\frac{AN}{BN} = \frac{AC + AD}{BD + BC}$$

Applying the bisector angle theorem, we have

$$\frac{AN}{BN} = \frac{AP}{BP}$$

This completes the proof.

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**MC–3.** Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function on $\mathbb{R}$. A point $x$ is called a *shadow point* of $f$ if and only if there exists $y \in \mathbb{R}$, $y > x$ such that $f(y) > f(x)$. Suppose that all the points of the open interval $I = (a, b)$, $(a < b)$ are shadow points of $f$ and $a$ and $b$, are not shadow points of $f$.

Prove that

(i) $f(x) \leq f(b)$ for all $a < x < b$.

(ii) $f(a) = f(b)$.

(IMC – 2011)

**Solution.** (i) We argue by contradiction. Indeed, suppose that exists $x_1 \in (a, b)$ such that $f(x_1) > f(b)$. Since $b$ it is not a shadow point of $f$, then for all $y > b$ is $f(y) \leq f(b)$. That is, we have for all $y > b$,
\[ f(x_1) > f(b) \geq f(y) \] (1)

Since \( x_1 \) is a shadow point of \( f \), then exists a point \( x_2 > x_1 \) such that \( f(x_2) > f(x_1) \). By (1) is \( x_1 < x_2 < b \). Since \( x_2 \) is also a shadow point of \( f \), then exists \( x_3 \) such that \( x_2 < x_3 < b \) with \( f(x_3) > f(x_2) \). Carrying out this procedure we can build up an increasing sequence \( \{x_n\}_{n \geq 1} \) bounded by \( b \). So, \( \{x_n\}_{n \geq 1} \) is convergent. Let \( \lim_{n \to +\infty} x_n = x \). We claim that \( x = b \).

(i) Let \( \{x_n\}_{n \geq 1} \) be a sequence of points of \( (a, b) \) such that \( x_n < a \). If \( x_n < a \), then \( f(x_n) < f(a) \). By (i) is \( f(x_n) \leq f(b) \). So, \( f(a) = \lim_{n \to +\infty} f(x_n) \leq f(b) \). Now, we will see that \( f(a) = f(b) \). Indeed, if \( f(a) < f(b) \), then \( a \) will be a shadow point of \( f \). But \( a \) is not a shadow point of \( f \), so \( f(a) = f(b) \) and we are done.

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**MC–4.** Place \( n \) points on a circle and draw in all possible chord joining these points. If no three chord are concurrent, find (with proof) the number of disjoint regions created.

(IMAC – 2011)

**Solution.** First, we prove that if a convex region crossed by \( L \) lines with \( P \) interior points of intersection, then the number of disjoint regions created is \( R_L = L + P + 1 \). To prove the preceding claim, we argue by Mathematical Induction on \( L \). Let \( R \) be an arbitrary convex region in the plane. For each \( L \geq 0 \), let \( A(L) \) be the statement that for each \( P \in \{1, 2, \ldots, \binom{L}{2}\} \), if \( L \) lines
that cross \( R \), with \( P \) intersection points inside \( R \), then the number of disjoint regions created inside \( R \) is \( R_L = L + P + 1 \).

When no lines intersect \( R \), then \( P = 0 \), and so, \( R_0 = 0 + 0 + 1 = 1 \) and \( A(0) \) holds. Fix some \( K \geq 0 \) and suppose that \( A(K) \) holds for \( K \) lines and some \( P \geq 0 \) with \( R_K = K + P + 1 \) regions. Consider a collection \( C \) of \( K + 1 \) lines each crossing \( R \) (not just touching), choose some line \( \ell \in C \), and apply \( A(K) \) to \( C\setminus\{\ell\} \) with some \( P \) intersection points inside \( R \) and \( R_K = K + P + 1 \) regions. Let \( S \) be the number of lines intersecting \( \ell \) inside \( R \). Since one draws a \((K + 1)\)–st line \( \ell \), starting outside \( R \), a new region is created when \( \ell \) first crosses the border of \( R \), and whenever \( \ell \) crosses a line inside of \( R \). Hence the number of new regions is \( S + 1 \).

Hence, the number of regions determined by the \( K + 1 \) lines is, on account of \( A(K) \),

\[
R_{K+1} = R_K + S + 1 = (K + P + 1) + S + 1 = (K + 1) + (P + S) + 1,
\]

where \( P + S \) is the total number of intersection points inside \( R \). Therefore, \( A(K + 1) \) holds and by the PMI the claim is proven. Finally, since the circle is convex and any intersection point is determined by a unique 4–tuple of points, then there are \( P = \binom{n}{4} \) intersection points and \( L = \binom{n}{2} \) chords and the number of regions is \( R = \binom{n}{4} + \binom{n}{2} + 1 \).

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**MC–5.** Let \( a, b, c \in (0, +\infty) \) such that \( a + b + c = 1 \). Prove that

\[
\frac{a}{a^3 + b^2c + c^2b} + \frac{b}{b^3 + c^2a + a^2c} + \frac{c}{c^3 + a^2b + b^2a} \leq 1 + \frac{8}{27abc}
\]

(IMAC – 2011)

**Solution.** To prove the the inequality claimed, we will apply CBS inequality. Indeed, we have

\[
\left( \sum_{cyc} a \right)^2 = \left( \sum_{cyc} \sqrt{x} \sqrt{\frac{a^2}{x}} \right)^2 \leq \left( \sum_{cyc} x \right) \left( \sum_{cyc} \frac{a^2}{x} \right)
\]

Writing the preceding in the most convenient form

\[
\sum_{cyc} \frac{a^2}{x} \geq \left( \sum_{cyc} a \right)^2 / \left( \sum_{cyc} x \right)
\]
we have
\[ a^3 + b^2c + c^2b = \frac{a^2}{a} + \frac{b^2}{c} + \frac{c^2}{b} \geq \left( \sum_{\text{cyc}} \frac{a}{a} \right)^2 / \left( \sum_{\text{cyc}} \frac{1}{a} \right) \]

from which immediately follows
\[ \frac{a}{a^3 + b^2c + c^2b} = a \sum_{\text{cyc}} \frac{1}{a} \left( \sum_{\text{cyc}} a \right)^2 \]

So, on account of the preceding and the constrain, we have
\[ L \leq \left( \sum_{\text{cyc}} a \right) \left( \sum_{\text{cyc}} \frac{1}{a} \right) \left( \sum_{\text{cyc}} a \right)^2 \]
\[ = \left( \sum_{\text{cyc}} \frac{1}{a} \right) \sum_{\text{cyc}} \frac{1}{a} = \frac{ab + bc + ca}{abc}, \]

where
\[ L = \frac{a}{a^3 + b^2c + c^2b} + \frac{b}{b^3 + c^2a + a^2c} + \frac{c}{c^3 + a^2b + b^2a} \]

Now, applying Jensen’s inequality to the convex function \( f : (0, +\infty) \to \mathbb{R} \) defined by \( f(t) = t^3 \), we get \( 9(a^3 + b^3 + c^3) \geq (a + b + c)^3 \). Taking into account the well-known identity
\[ a^3 + b^3 + c^3 = (a + b + c)[(a^2 + b^2 + c^2) - (ab + bc + ca)] + 3abc \]
\[ = (a + b + c)[(a + b + c)^2 - 3(ab + bc + ca)] + 3abc \]

and again, on account of the constrain, we have
\[ a^3 + b^3 + c^3 = 1 - 3(ab + bc + ca)] + 3abc; \]

and \( 9(a^3 + b^3 + c^3) \geq (a + b + c)^3 \) becomes \( 27(abc - (ab + bc + ca)) + 8 \geq 0 \) from which follows \( ab + bc + ca \leq \frac{8}{27} + abc \). Finally, we have
\[ \frac{a}{a^3 + b^2c + c^2b} + \frac{b}{b^3 + c^2a + a^2c} + \frac{c}{c^3 + a^2b + b^2a} \leq \frac{ab + bc + ca}{abc} \leq 1 + \frac{8}{27abc} \]

Equality holds when \( a = b = c = 1/3 \) and we are done.
MC–6. Let $m$ and $n$ be distinct integer numbers. Find all functions $f : \mathbb{Z} \to \mathbb{R}$ such that $f(mx + ny) = mf(x) + nf(y)$ for all $x, y \in \mathbb{Z}$.

(IMAC – 2009)

Solution. We distinguish two cases. First, we suppose that $m + n \neq 1$. Then, for $x = y = 0$ we obtain $f(0) = (m + n)f(0)$ from which follows $f(0) = 0$. For $y = 0$ we get $f(mx)mf(x)$ for all $x \in \mathbb{Z}$. For $x = 0$ we get $f(ny) = nf(y)$ for all $y \in \mathbb{Z}$. Now we have $f(mx + my) = mf(nx + nf(my) = mn(f(x) + f(y))$, for all $x, y \in \mathbb{Z}$ and $f(x + y) = f(x) + f(y)$ or $f(kx) = kf(x)$ for all $k, x \in \mathbb{Z}$ from which follows that $f(x) = cx$, $x \in \mathbb{Z}$ and $c \in \mathbb{R}$ is solution of the claimed functional equation.

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MC–7. Let $a, b, c$ be three positive real numbers such that $a + b + c + \sqrt{abc} = 2$. Prove that $a + b + c + \sqrt{abc} = 2$.

(Ibero Longlist – 2009)

Solution. Putting $A = \sqrt{\frac{bc}{a}}, B = \sqrt{\frac{ca}{b}}$, and $A = \sqrt{\frac{ab}{c}}$, the constrain becomes $ABC + AB + BC + CA = 2$ and the inequality claimed can be written as

$$\frac{1}{(A + 1)^2 + 1} + \frac{1}{(B + 1)^2 + 1} + \frac{1}{(C + 1)^2 + 1} \geq \frac{3}{4}$$

Since,

$$ABC + AB + BC + CA = 2 \iff (A+1)(B+1)(C+1) = (A+1)+(B+1)+(C+1),$$
then exist three numbers $\alpha, \beta, \gamma \in [0, \pi]$ such that $A + 1 = \tan \alpha, B + 1 = \tan \beta, C + 1 = \tan \gamma$, and $\alpha + \beta + \gamma = \pi$. Since $A > 0$, then $\tan \alpha > 1$ and $\alpha \in (\pi/4, \pi/2)$. Likewise, $\beta \in (\pi/4, \pi/2)$ and $\gamma \in (\pi/4, \pi/2)$.

Moreover, we have
\[
\frac{1}{(A + 1)^2 + 1} + \frac{1}{(B + 1)^2 + 1} + \frac{1}{(C + 1)^2 + 1} = \frac{1}{1 + \tan^2 \alpha} + \frac{1}{1 + \tan^2 \beta} + \frac{1}{1 + \tan^2 \gamma} = \cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma
\]

Now we consider the function $f : (\pi/4, \pi/2) \to \mathbb{R}$ defined by $f(x) = \cos^2 x$.
Since, $f'(x) = -\sin 2x$ and $f''(x) = -2 \cos 2x > 0$ for all $x \in (\pi/4, \pi/2)$, then $f$ is convex in $(\pi/4, \pi/2)$. Applying Jensen’s inequality, yields
\[
\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = f(\alpha) + f(\beta) + f(\gamma) \geq 3f\left(\frac{\alpha + \beta + \gamma}{3}\right) = \frac{3}{4}
\]
Equality holds when $\alpha = \beta = \gamma = \pi/3$. That is, when $A = B = C = \sqrt{3} - 1$ or $a = b = c = 4 - 2\sqrt{3}$.

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**MC–8.** Let $a, b$ be positive real numbers and let $f : [a, b] \to \mathbb{R}$ be a continuous function. Prove that there exits $c \in (a, b)$ such that
\[
\frac{1}{2} f(c) = \left(\frac{c}{a^2 - c^2} + \frac{c}{b^2 - c^2}\right) \int_a^c f(t) \, dt
\]

(Spanish training for IMC – 2005)

**Solution.** Consider the function $F : [a, b] \to \mathbb{R}$ defined by
\[
F(x) = (x^2 - a^2)(x^2 - b^2) \int_a^x f(t) \, dt.
\]
The function $F(x)$ is continuous in $[a, b]$, derivable in $(a, b)$ and $F(a) = F(b) = 0$. Therefore, according to Rolle’s theorem, we have that there exists $c \in (a, b)$ such that $F'(c) = 0$. That is,
\[
2c(c^2 - b^2) \int_a^c f(t) \, dt + 2c(c^2 - a^2) \int_a^c f(t) \, dt + (c^2 - a^2)(c^2 - b^2)f(c) = 0
\]
or equivalently
\[ 2c \left[ (c^2 - a^2) + (c^2 - b^2) \right] \int_a^c f(t) \, dt + (c^2 - a^2)(c^2 - b^2)f(c) = 0. \]

Dividing both sides by \((c^2 - a^2)(c^2 - b^2)\), we get
\[ \left( \frac{2c}{c^2 - a^2} + \frac{2c}{c^2 - b^2} \right) \int_a^c f(t) \, dt + f(c) = 0 \]
and we are done.

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**MC–9.** Let \( A(z) = \sum_{k=0}^{n} a_k z^k \) be a polynomial of degree \( n \) with complex coefficients having all its zeros in the disk \( C = \{ z \in \mathbb{C} : |z| \leq \sqrt{6} \} \). Show that
\[ |A(3z)| \geq \left( \frac{3 + \sqrt{6}}{2 + \sqrt{6}} \right)^n |A(2z)| \]
for any complex number \( z \) with \(|z| = 1\).

(IMC Longlist – 2009)

**Solution.** Let \( z_1, z_2, \ldots, z_n \) be the zeros (not necessarily distinct) of \( A(z) \). Then \( z_k = r_k e^{i \theta_k}, 1 \leq k \leq n, 0 \leq \theta < 2\pi \) and \( r_k \leq \sqrt{6} \). So, we can write
\[ A(z) = C \prod_{k=1}^{n} (z - z_k) = C \prod_{k=1}^{n} (z - r_k e^{i \theta_k}) \]

For any \( z = e^{i \theta}, 0 \leq \theta < 2\pi \), we have
\[ \frac{|A(2z)|}{|A(3z)|} = \frac{|A(2e^{i \theta})|}{|A(3e^{i \theta})|} = \prod_{k=1}^{n} \left| \frac{2e^{i \theta} - r_k e^{i \theta_j}}{3e^{i \theta} - r_k e^{i \theta_j}} \right| = \prod_{k=1}^{n} \left| \frac{2e^{i(\theta - \theta_k)} - r_k}{3e^{i(\theta - \theta_k)} - r_k} \right| \]

On the other hand, for \( 1 \leq k \leq n \), we have
\[ \left| \frac{2e^{i(\theta - \theta_k)} - r_k}{3e^{i(\theta - \theta_k)} - r_k} \right|^2 = \left( \frac{2e^{i(\theta - \theta_k)} - r_k}{3e^{i(\theta - \theta_k)} - r_k} \right) \left( \frac{2e^{i(\theta - \theta_k)} - r_k}{3e^{i(\theta - \theta_k)} - r_k} \right) = \frac{4 + r_k^2 - 4r_k \cos(\theta - \theta_k)}{9 + r_k^2 - 6r_k \cos(\theta - \theta_k)} \leq \left( \frac{2 + r_k}{3 + r_k} \right)^2 \]
Indeed, the last inequality
\[
\frac{4 + r_k^2 - 4r_k \cos(\theta - \theta_k)}{9 + r_k^2 - 6r_k \cos(\theta - \theta_k)} \leq \frac{4 + r_k^2 + 4r_k}{9 + r_k^2 + 6r_k}
\]
is equivalent to
\[
12r_k (1 + \cos(\theta - \theta_k)) \geq 2r_k^3 (1 + \cos(\theta - \theta_k)),
\]
or \(6 \geq r_k^2\) which is true because \(r_k \leq \sqrt{6}, 1 \leq k \leq n\). Now, immediately follows that
\[
\left| \frac{A(2z)}{A(3z)} \right| \leq \prod_{k=1}^{n} \left( \frac{2 + r_k}{3 + r_k} \right) \leq \prod_{k=1}^{n} \left( \frac{2 + \sqrt{6}}{3 + \sqrt{6}} \right) = \left( \frac{2 + \sqrt{6}}{3 + \sqrt{6}} \right)^n
\]
Equality holds for the polynomial \(A(z) = (z + \sqrt{6})^n\) and we are done.

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**MC–10.** Compute
\[
\lim_{n \to \infty} \ln \left[ \frac{1}{2^n} \prod_{k=1}^{n} \left( 2 + \frac{k}{n^2} \right) \right]
\]
(Spanish Training for IMC – 2009)

**Solution 1.** It is easy to see that
\[
\prod_{k=1}^{n} \left( 2 + \frac{k}{n^2} \right) = \left( 2 + \frac{1}{n^2} \right) \left( 2 + \frac{2}{n^2} \right) \cdots \left( 2 + \frac{n}{n^2} \right) = \frac{(2n^2 + 1)(2n^2 + 2) \cdots (2n^2 + n)}{n^{2n}} = \frac{\Gamma(2n^2 + n + 1)}{n^{2n} \Gamma(2n^2 + 1)},
\]
where \(\Gamma\) denotes the gamma function. From the asymptotic expansion:
\[
\ln \Gamma(x) = \left( x - \frac{1}{2} \right) \ln x - x + \ln \sqrt{2\pi} + \frac{1}{12x} + O(x^{-3}) \quad (x \to \infty)
\]
(see [1] p. 257, 6.1.41), we find that

\[
\ln \left[ \frac{1}{2^n} \prod_{k=1}^{n} \left( 2 + \frac{k}{n^2} \right) \right] = \ln \left[ \frac{\Gamma(2n^2 + n + 1)}{2^n n^{2n} \Gamma(2n^2 + 1)} \right] \\
= \ln \Gamma(2n^2 + n + 1) - \ln \Gamma(2n^2 + 1) - n \ln 2 - 2n \ln n \\
\sim \frac{1}{4} + \frac{5}{24n} + O \left( \frac{1}{n^2} \right) \quad (n \to \infty).
\]

Hence,

\[
\lim_{n \to \infty} \ln \left[ \frac{1}{2^n} \prod_{k=1}^{n} \left( 2 + \frac{k}{n^2} \right) \right] = \frac{1}{4}.
\]


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**Solution 2.** We have that

\[
\ln \left[ \frac{1}{2^n} \prod_{k=1}^{n} \left( 2 + \frac{k}{n^2} \right) \right] = \sum_{k=1}^{n} \ln \left( 1 + \frac{k}{2n^2} \right),
\]

and since

\[
x - \frac{x^2}{2} < \ln(1 + x) < x \quad \forall x > 0,
\]

then

\[
\sum_{k=1}^{n} \frac{k}{2n^2} - \frac{1}{2} \sum_{k=1}^{n} \frac{k^2}{4n^4} < \ln \left[ \frac{1}{2^n} \prod_{k=1}^{n} \left( 2 + \frac{k}{n^2} \right) \right] < \sum_{k=1}^{n} \frac{k}{2n^2}.
\]

But

\[
\sum_{k=1}^{n} \frac{k}{2n^2} = \frac{1}{2n^2} \sum_{k=1}^{n} k = \frac{n(n + 1)}{4n^2} \to \frac{1}{4}
\]

and

\[
\sum_{k=1}^{n} \frac{k^2}{4n^4} = \frac{1}{4n^2} \sum_{k=1}^{n} k^2 = \frac{n(n + 1)(2n + 1)}{24n^4} \to 0.
\]
so
\[
\ln \left[ \frac{1}{2^n} \prod_{k=1}^{n} \left( 2 + \frac{k}{n^2} \right) \right] \longrightarrow \frac{1}{4}
\]

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\textbf{Solution 3.} We begin with a lemma.

**Lemma.** Let \( f : I \subseteq \mathbb{R} \to \mathbb{R} \) and let \( a \in I \). If \( f \) is derivable in \( a \) then the sequence \( \{x_n\}_{n \geq 1} \) defined by
\[
x_n = \sum_{k=1}^{n} f \left( a + \frac{k}{n^2} \right) - nf(a)
\]
is convergent and \( \lim_{n \to \infty} x_n = \frac{1}{2} f'(a) \).

**Proof.** Since \( f \) is derivable in \( a \), then \( f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \). This means, as it is well known, that for all \( \varepsilon > 0 \) there exists \( \delta = \delta(\varepsilon) > 0 \) such that for all \( x \in (a - \delta, a + \delta) \), \( x \neq a \) or \( 0 < |x - a| < \delta \) is
\[
\left| \frac{f(x) - f(a)}{x - a} - f'(a) \right| < \varepsilon \iff f'(a) - \varepsilon < \frac{f(x) - f(a)}{x - a} < f'(a) + \varepsilon
\]
Let \( x = a + \frac{k}{n^2} \), \( 1 \leq k \leq n \) then there exists \( n_1 \in \mathbb{N} \) such that for all \( n \geq n_1 \) we have \( 0 < |x - a| = \frac{k}{n^2} \leq \frac{1}{n} \). Thus, for all \( k \in \{1, 2, \ldots, n\} \), we have
\[
(f'(a) - \varepsilon) \frac{k}{n^2} < f \left( a + \frac{k}{n^2} \right) - f(a) < (f'(a) + \varepsilon) \frac{k}{n^2}
\]
Adding up the preceding inequalities, we get
\[
(f'(a) - \varepsilon) \frac{n+1}{2n} < \sum_{k=1}^{n} f \left( a + \frac{k}{n^2} \right) - nf(a) < (f'(a) + \varepsilon) \frac{n+1}{2n}
\]
or
\[
(f'(a) - \varepsilon) \frac{n+1}{2n} < x_n < (f'(a) + \varepsilon) \frac{n+1}{2n} \iff \frac{f'(a) - \varepsilon (n+1)}{2n} < x_n - \frac{1}{2} f'(a) < \frac{f'(a) + \varepsilon (n+1)}{2n}.
\]
Since \( \lim_{n \to \infty} \frac{f'(a)}{2n} = 0 \), then immediately follows that
\[
\lim_{n \to \infty} x_n = \frac{1}{2} f'(a)
\]

Now, we set \( f : (-1, +\infty) \to \mathbb{R} \) defined by \( f(x) = \ln(1 + x) \) and \( a = 1 \) into the previous lemma, and we get
\[
x_n = \sum_{k=1}^{n} \ln \left( 2 + \frac{k}{n^2} \right) - n \ln 2 = \ln \left[ \frac{1}{2^n} \prod_{k=1}^{n} \left( 2 + \frac{k}{n^2} \right) \right]
\]

Since \( f'(x) = \frac{1}{1 + x} \) then \( f'(1) = \frac{1}{2} \) and
\[
\lim_{n \to \infty} \ln \left[ \frac{1}{2^n} \prod_{k=1}^{n} \left( 2 + \frac{k}{n^2} \right) \right] = \frac{1}{2} f'(1) = \frac{1}{4}
\]

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Also solved by Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain, José Gibergans-Báguena, Barcelona Tech, Barcelona, Spain.

MC–11. Let \( a, b \) be positive integers. Prove that
\[
\frac{\varphi(ab)}{\sqrt{\varphi^2(a^2) + \varphi^2(b^2)}} \leq \frac{\sqrt{2}}{2},
\]
where \( \varphi(n) \) is the Euler’s totient function.

(Spanish Training for IMC – 2009)

**Solution.** Let \( a = \prod_{i=1}^{p} p_i^{a_i} \prod_{j=1}^{q} q_j^{b_j} \) and \( b = \prod_{i=1}^{p} p_i^{a_i} \prod_{k=1}^{r} r_k^{c_k} \) being \( a_i, b_j \) and \( c_k \) nonnegative integers. Then,
\[
\varphi(a) = a \prod_{i=1}^{p} \left( 1 - \frac{1}{p_i} \right) \prod_{j=1}^{q} \left( 1 - \frac{1}{q_j} \right), \quad \varphi(b) = b \prod_{i=1}^{p} \left( 1 - \frac{1}{p_i} \right) \prod_{k=1}^{r} \left( 1 - \frac{1}{r_k} \right)
\]

On the other hand, \( \varphi(ab) = ab \prod_{i=1}^{p} \left( 1 - \frac{1}{p_i} \right) \prod_{j=1}^{q} \left( 1 - \frac{1}{q_j} \right) \prod_{k=1}^{r} \left( 1 - \frac{1}{r_k} \right) \) and
\[
\varphi(a^2) = a^2 \prod_{i=1}^{p} \left( 1 - \frac{1}{p_i} \right) \prod_{j=1}^{q} \left( 1 - \frac{1}{q_j} \right), \quad \varphi(b^2) = b^2 \prod_{i=1}^{p} \left( 1 - \frac{1}{p_i} \right) \prod_{k=1}^{r} \left( 1 - \frac{1}{r_k} \right)
\]
Since \( \prod_{j=1}^{q} \left( 1 - \frac{1}{q_j} \right) \prod_{k=1}^{r} \left( 1 - \frac{1}{r_k} \right) \leq 1 \), then

\[
\prod_{j=1}^{q} \left( 1 - \frac{1}{q_j} \right) \prod_{k=1}^{r} \left( 1 - \frac{1}{r_k} \right) \leq \sqrt{\prod_{j=1}^{q} \left( 1 - \frac{1}{q_j} \right) \prod_{k=1}^{r} \left( 1 - \frac{1}{r_k} \right)}
\]

Taking into account GM-QM inequality, we have

\[
\varphi(ab) \leq \sqrt{\varphi(a^2) \varphi(b^2)} \leq \sqrt{\frac{\varphi^2(a^2) + \varphi^2(b^2)}{2}}
\]

from which the statement follows.

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**MC–12.** If \( x_i > 0 \) and \( \alpha_i \in \mathbb{R} \), then

\[
V_n(x, \alpha) = \begin{vmatrix}
  x_1^{\alpha_1} & x_2^{\alpha_1} & \cdots & x_n^{\alpha_1} \\
  x_1^{\alpha_2} & x_2^{\alpha_2} & \cdots & x_n^{\alpha_2} \\
  \vdots & \vdots & \ddots & \vdots \\
  x_1^{\alpha_n} & x_2^{\alpha_n} & \cdots & x_n^{\alpha_n}
\end{vmatrix}
\]

is called a generalized Vandermonde determinant of order \( n \). Let \( 0 < x_1 < x_2 < \ldots < x_n \), and \( \alpha_1 < \alpha_2 < \ldots < \alpha_n \) be real numbers. Prove that \( V_n(x, \alpha) > 0 \).

(Spanish Training for IMC – 2008)

**Solution.** We will argue by mathematical induction. The case when \( n = 1 \) trivially holds. For \( n = 2 \), using Lagrange’s mean value theorem, we have

\[
V_2(x, \alpha) = \begin{vmatrix}
  x_1^{\alpha_1} & x_2^{\alpha_1} \\
  x_1^{\alpha_2} & x_2^{\alpha_2}
\end{vmatrix} = x_1^{\alpha_1} x_2^{\alpha_2} - x_1^{\alpha_2} x_2^{\alpha_1}
\]

\[
= x_1^{\alpha_1} x_2^{\alpha_1} (x_2^{\alpha_2-\alpha_1} x_1^{\alpha_2-\alpha_1}) = x_1^{\alpha_1} x_2^{\alpha_1} (x_2 - x_1) \left. \frac{dt^{\alpha_2-\alpha_1}}{dt} \right|_{t=\xi}
\]

\[
= x_1^{\alpha_1} x_2^{\alpha_1} (x_2 - x_1)(\alpha_2 - \alpha_1) \xi^{\alpha_2-\alpha_1-1} > 0,
\]
where \( x_1 < \xi < x_2 \). Suppose that the inequality holds for \( n - 1 \) with \( n \geq 2 \).

Namely,

\[
V_{n-1}(y, \alpha) = \begin{vmatrix}
y_2^{\alpha_1} y_3^{\alpha_1} \cdots y_n^{\alpha_1} \\
y_2^{\alpha_2} y_3^{\alpha_2} \cdots y_n^{\alpha_2} \\
\vdots & \vdots & \ddots & \vdots \\
y_2^{\alpha_n} y_3^{\alpha_n} \cdots y_n^{\alpha_n}
\end{vmatrix} > 0,
\]

where \( 0 < y_2 < y_3 < \ldots < y_n \) and \( \alpha_2 < \alpha_3 < \ldots < \alpha_n \). Let

\[
f(x_n) = \begin{vmatrix} 1 & 1 & \cdots & 1 \\
x_1^{\alpha_2-\alpha_1} x_2^{\alpha_2-\alpha_1} \cdots x_n^{\alpha_2-\alpha_1} \\
\vdots & \vdots & \ddots & \vdots \\
x_1^{\alpha_n-\alpha_1} x_2^{\alpha_n-\alpha_1} \cdots x_n^{\alpha_n-\alpha_1}
\end{vmatrix}
\]

Applying Lagrange’s mean value theorem again, we get

\[
f(x_n) = f(x_n) - f(x_{n-1}) = (x_n - x_{n-1})f'(\xi_n)
\]

\[
= (x_n - x_{n-1}) \begin{vmatrix} 1 & 1 & \cdots & 1 \\
x_1^{\alpha_2-\alpha_1} x_2^{\alpha_2-\alpha_1} \cdots x_n^{\alpha_2-\alpha_1} \\
\vdots & \vdots & \ddots & \vdots \\
x_1^{\alpha_n-\alpha_1} x_2^{\alpha_n-\alpha_1} \cdots x_n^{\alpha_n-\alpha_1}
\end{vmatrix}
\]

\[
= (x_n - x_{n-1}) \begin{vmatrix} 1 & 1 & \cdots & 1 & 0 \\
x_1^{\alpha_2-\alpha_1} x_2^{\alpha_2-\alpha_1} \cdots x_n^{\alpha_2-\alpha_1} & (\alpha_2 - \alpha_1)\xi_n^{\alpha_2-\alpha_1-1} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
x_1^{\alpha_n-\alpha_1} x_2^{\alpha_n-\alpha_1} \cdots x_n^{\alpha_n-\alpha_1} & (\alpha_n - \alpha_1)\xi_n^{\alpha_n-\alpha_1-1}
\end{vmatrix}
\]

where \( x_{n-1} < \xi_n < x_n \). Applying the same procedure, it is easy to obtain

\[
f(x_n) = \prod_{j=k-1}^{n} (x_j - x_{j-1}) \begin{vmatrix} 1 & 1 & \cdots & 0 \\
x_1^{\alpha_2-\alpha_1} (\alpha_2 - \alpha_1)\xi_2^{\alpha_2-\alpha_1-1} \cdots (\alpha_2 - \alpha_1)\xi_n^{\alpha_2-\alpha_1-1} \\
\vdots & \vdots & \ddots & \vdots \\
x_1^{\alpha_n-\alpha_1} (\alpha_n - \alpha_1)\xi_n^{\alpha_n-\alpha_1-1} \cdots (\alpha_n - \alpha_1)\xi_n^{\alpha_n-\alpha_1-1}
\end{vmatrix}
\]

\[
= \prod_{j=2}^{n} (x_j - x_{j-1}) \prod_{i=2}^{n} (\alpha_i - \alpha_1) \begin{vmatrix} \xi_2^{\alpha_2-\alpha_1-1} \xi_3^{\alpha_2-\alpha_1-1} \cdots \xi_n^{\alpha_2-\alpha_1-1} \\
\xi_2^{\alpha_3-\alpha_1-1} \xi_3^{\alpha_3-\alpha_1-1} \cdots \xi_n^{\alpha_3-\alpha_1-1} \\
\vdots & \vdots & \ddots & \vdots \\
\xi_2^{\alpha_n-\alpha_1-1} \xi_3^{\alpha_n-\alpha_1-1} \cdots \xi_n^{\alpha_n-\alpha_1-1}
\end{vmatrix}
\]

\[
= \prod_{j=2}^{n} [(\alpha_j - \alpha_1)(x_j - x_{j-1})\xi^{-(\alpha_j+1)}] V_{n-1}(\xi, \alpha),
\]
where $0 < x_1 < x_2 < \ldots < x_{n-1} < \xi_n < x_n$ and $\alpha_2 < \alpha_3 < \ldots < \alpha_n$. For general $n$ from the preceding, we have

$$V_n(x,\alpha) = f(x_n)\prod_{j=1}^{n} x_j^{\alpha_1}$$

$$= V_{n-1}(\xi,\alpha)\prod_{j=1}^{n} x_j^{\alpha_1} \prod_{j=2}^{n} (\alpha_j - \alpha_1)(x_j - x_{j-1})\xi^{-(\alpha_1+1)} > 0,$$

where $0 < x_1 < \xi_1 < x_2 < \ldots < x_{n-1} < \xi_n < x_n$ and $\alpha_1 < \alpha_2 < \ldots < \alpha_n$. Therefore, by the PMI the statement is proved.

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Also solved by José Gibergans-Báguena, Barcelona Tech, Barcelona, Spain.

**MC–13.** Let $X$ be a set of cardinal $n$. Determine

$$\sum_{(A,B) \subseteq X \times X} \text{card}(A \cap B)$$

(Spanish Training for IMC – 2008)

**Solution.** Given $A \subseteq X$ and an integer $i$, let us calculate the number of ways to choose $B \subseteq X$ such that

$$\text{card}(A \cap B) = i.$$ 

Let $k$ be the number of elements of $A$. The set $B$ must have $i$ elements in $A$ and some other elements not in $A$. Then, to calculate the number of ways to choose $B$, we have $\binom{k}{i}$ ways to choose $i$ elements from $A$, and $2^{n-k}$ ways to choose the others that are not in $A$. Hence, we have

$$2^{n-k}\binom{k}{i}$$

ways to choose $B$. Therefore, given $A \subseteq X$ such that $\text{card}(A) = k$,

$$\sum_{B \subseteq X} \text{card}(A \cap B) = \sum_{i=0}^{k} i 2^{n-k}\binom{k}{i} = 2^{n-k}\sum_{i=0}^{k} i \binom{k}{i}.$$
Now, for each $k$, we can choose $A$ in $\binom{n}{k}$ different ways, and then,

$$\sum_{(A,B) \subseteq X \times X} \text{card}(A \cap B) = \sum_{A \subseteq X} \sum_{B \subseteq X} \text{card}(A \cap B) = \sum_{k=0}^{n} \binom{n}{k} \sum_{B \subseteq X} \text{card}(A \cap B) = \sum_{k=0}^{n} \binom{n}{k} 2^{n-k} \sum_{i=1}^{k} \binom{k}{i}.$$

Now, taking derivatives in both sides of the well known equality

$$(1 + x)^k = \sum_{i=0}^{n} x^i \binom{k}{i},$$

we get that

$$k(1 + x)^{k-1} = \sum_{i=1}^{n} ix^{i-1} \binom{k}{i},$$

and setting $x = 1$,

$$\sum_{i=1}^{k} i \binom{k}{i} = k2^{k-1}.$$ 

Hence,

$$\sum_{k=0}^{n} \binom{n}{k} 2^{n-k} \sum_{i=1}^{k} \binom{k}{i} = \sum_{k=0}^{n} \binom{n}{k} 2^{n-k} k2^{k-1} = 2^{n-1} \sum_{k=0}^{n} \binom{n}{k} = 2^{n-1} n2^{n-1},$$

and then,

$$\sum_{(A,B) \subseteq X \times X} \text{card}(A \cap B) = n4^{n-1}.$$ 

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MC–14. Compute

$$\int_{1}^{\infty} \frac{dt}{2[t] + 3[t]^2 + [t]^3},$$

where $[x]$ represents the integer part of $x$.

(Spanish Training Team for IMC – 2008)
Solution 1.

\[
\int_1^{\infty} \frac{dt}{2[t] + 3[t]^2 + [t]^3} = \lim_{N \to \infty} \int_1^{N} \frac{dt}{2[t] + 3[t]^2 + [t]^3}
\]

\[
= \lim_{N \to \infty} \sum_{k=1}^{N-1} \int_{k+1}^{N} \frac{dt}{2[t] + 3[t]^2 + [t]^3}
\]

\[
= \lim_{N \to \infty} \sum_{k=1}^{N-1} \frac{1}{2k + 3k^2 + k^3}
\]

\[
= \lim_{N \to \infty} \sum_{k=1}^{N-1} \left( \frac{1}{2} \frac{1}{k} - \frac{1}{k+1} + \frac{1}{k+2} \right)
\]

\[
= \lim_{N \to \infty} \frac{1}{2} - \frac{1}{4} - \frac{1/2}{N} + \frac{1/2}{N+1} = \frac{1}{4}.
\]

Where we have used that \([t] = k\) for \(t \in [k, k+1)\), and the decomposition into simple fractions of \(1/(2k + 3k^2 + k^3)\).

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Solution 2. Let \(f(t) = \frac{1}{2[t] + 3[t]^2 + [t]^3}\). If \(n \leq t < n + 1\), the \([t] = n\) and

\[
\int_n^{n+1} f(t) \, dt = \int_n^{n+1} \frac{dt}{2n + 3n^2 + n^3}
\]

\[
= \frac{1}{n(n+1)(n+2)} \int_n^{n+1} dt = \frac{1}{n(n+1)(n+2)}
\]

Therefore,

\[
\int_1^{n} f(t) \, dt = \sum_{k=1}^{n-1} \int_k^{k+1} f(t) \, dt = \sum_{k=1}^{n-1} \frac{1}{k(k+1)(k+2)} = \frac{1}{4} - \frac{1}{2n(n+1)}
\]

Taking limits when \(n \to \infty\), we have

\[
\int_1^{\infty} \frac{dt}{2[t] + 3[t]^2 + [t]^3} = \lim_{n \to \infty} \int_1^{n} f(t) \, dt
\]

\[
= \lim_{n \to \infty} \left( \frac{1}{4} - \frac{1}{2n(n+1)} \right) = \frac{1}{4}
\]

and we are done.
Find all triplets \((x, y, z)\) of real numbers such that
\[
\begin{align*}
x^2 + \sqrt{y^2 + 12} &= \sqrt{y^2 + 60}, \\
y^2 + \sqrt{z^2 + 12} &= \sqrt{z^2 + 60}, \\
z^2 + \sqrt{x^2 + 12} &= \sqrt{x^2 + 60}.
\end{align*}
\]

(Spanish First Stage Contest – 2009)

**Solution.** Putting \(x^2 = a\), \(y^2 = b\) and \(z^2 = c\), yields
\[
\begin{align*}
a + \sqrt{b + 12} &= \sqrt{b + 60}, \\
b + \sqrt{c + 12} &= \sqrt{c + 60}, \\
c + \sqrt{a + 12} &= \sqrt{a + 60}.
\end{align*}
\]
\[
\Rightarrow \quad \begin{align*}
a &= \sqrt{b + 60} - \sqrt{b + 12}, \\
b &= \sqrt{c + 60} - \sqrt{c + 12}, \\
c &= \sqrt{a + 60} - \sqrt{a + 12}.
\end{align*}
\]

Now we consider the function \(f : [0, +\infty) \to \mathbb{R}\) defined by
\[
f(t) = \sqrt{t + 60} - \sqrt{t + 12} = \frac{48}{\sqrt{t + 60} + \sqrt{t + 12}}.
\]
Since \(0 \leq u < v\) is
\[
\frac{48}{\sqrt{u + 60} + \sqrt{u + 12}} > \frac{48}{\sqrt{v + 60} + \sqrt{v + 12}}
\]
then \(f\) is decreasing and the same occurs with \(f(f(t))\). On the other hand, from \(f(b) = a\), \(f(c) = b\) and \(f(a) = c\) we get \(f(f(f(a))) = a\) which is possible if and only if \(f(a) = a\). Next, we find the fixed points of \(f\). That is, we have to solve the equation
\[
\sqrt{t + 60} - \sqrt{t + 12} = t \Leftrightarrow 48 - t^2 = 2t\sqrt{t + 12}
\]
Since \(48 - t^2 \geq 0\) then \(t \in [0, 4\sqrt{3}]\) from which follows
\[
t^4 - 4t^3 - 144t^2 + 2304 = (t - 4)(t^3 - 144t - 576) = 0
\]
Since for all \(t \in [0, 4\sqrt{3}]\) is \(t^3 < 144t + 576\), then \(t = 4\) is the only fixed point of \(f\) and the solutions are \((a, b, c) = (4, 4, 4)\). Taking into account that \(x^2 = a, y^2 = b, z^2 = c\), we get that the solutions of the system are
\[
(2, 2, 2), \quad (-2, 2, 2), \quad (2, -2, 2), \quad (2, 2, -2), \\
(-2, -2, 2), \quad (-2, 2, -2), \quad (2, -2, -2), \quad (-2, -2, -2).
\]
MC–16. Let $p$ and $q$ be two prime numbers and let $r$ be a whole number. Find all possible values of $p, q, r$ for which
\[
\frac{1}{p + 1} + \frac{1}{q + 1} - \frac{1}{(p + 1)(q + 1)} = \frac{1}{r}
\]

Solution. We have,
\[
\frac{1}{p + 1} + \frac{1}{q + 1} - \frac{1}{(p + 1)(q + 1)} = \frac{p + q + 1}{(p + 1)(q + 1)}
\]
The fraction $\frac{p + q + 1}{(p + 1)(q + 1)}$ will be equal to $\frac{1}{r}$ with $r$ integer, when
\[
p + q + 1|(p + 1)(q + 1) \iff p + q + 1|(p + 1)(q + 1) - (p + q + 1) \iff p + q + 1|pq
\]
Since the only divisors of $pq$ are $1, p, q, pq$, then $p + q + 1|pq$ in the following cases: (1) $p + q + 1 = 1$ which is impossible; (2) $p + q + 1 = p$ which is also impossible; (3) $p + q + 1 = q$ impossible; and (4) $p + q + 1 = pq$.
In the last case we have
\[
pq - p - q - 1 = 0 \iff pq - p - q + 1 = 2 \iff (p - 1)(q - 1) = 2
\]
The only solutions of the preceding equation are $(2, 3)$ and $(3, 2)$, and therefore, the unique numbers that satisfy the statement are $p = 2, q = 3$ and $r = 2$, and also $p = 3, q = 2$ and $r = 2$.

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Also solved by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain.

MC–17. Solve in $\mathbb{R}$ the following system of equations:
\[
\begin{align*}
\sqrt{x} + \sqrt{y} + \sqrt{z} &= 3 \\
x\sqrt{x} + y\sqrt{y} + z\sqrt{z} &= 3 \\
x^2\sqrt{x} + y^2\sqrt{y} + z^2\sqrt{z} &= 3
\end{align*}
\]
Solution. Applying Cauchy’s inequality \((\vec{u} \cdot \vec{v})^2 \leq ||\vec{u}||^2 ||\vec{v}||^2\) to the vectors \(\vec{u} = (\sqrt{x}, \sqrt{y}, \sqrt{z})\) and \(\vec{v} = (x \sqrt{x}, y \sqrt{y}, z \sqrt{z})\), we get
\[
(x \sqrt{x} + y \sqrt{y} + z \sqrt{z})^2 \leq (\sqrt{x} + \sqrt{y} + \sqrt{z})(x^2 \sqrt{x} + y^2 \sqrt{y} + z^2 \sqrt{z})
\]
or \(3^2 \leq 3 \cdot 3\). Equality holds when vectors \(\vec{u}\) and \(\vec{v}\) are collinear. That is, when
\[
\frac{x \sqrt{x}}{\sqrt{x}} = \frac{y \sqrt{y}}{\sqrt{y}} = \frac{z \sqrt{z}}{\sqrt{z}} = k
\]
or, \(x = y = z = k\). Then, from \(\sqrt{x} + \sqrt{y} + \sqrt{z} = 3\), we get \(3 \sqrt{k} = 3\). That is, \(k = 1\) and the only solution is \((x, y, z) = (1, 1, 1)\).

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Also solved by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain, Oscar Rivero Salgado, Barcelona Tech, Barcelona, Spain.

MC–18. Find the biggest positive integer that can not be written in the form \(5a + 503b\) where \(a\) are \(b\) nonnegative integer numbers.

(IMAC Longlist – 2007)

Solution. Let \(A = \{5a + 503b \mid a, b \in \mathbb{Z}^+\}\). We have to find the biggest positive integer not belonging to the set \(A\). To do it, we imagine the positive integers written in an array with five rows and infinite columns like the following:

<table>
<thead>
<tr>
<th></th>
<th>C1</th>
<th>C2</th>
<th>C101</th>
<th>C201</th>
<th>C302</th>
<th>C402</th>
<th>C403</th>
</tr>
</thead>
<tbody>
<tr>
<td>F1</td>
<td>1</td>
<td>6</td>
<td>...</td>
<td>501</td>
<td>...</td>
<td>1506</td>
<td>...</td>
</tr>
<tr>
<td>F2</td>
<td>2</td>
<td>7</td>
<td>...</td>
<td>502</td>
<td>...</td>
<td>1507</td>
<td>...</td>
</tr>
<tr>
<td>F3</td>
<td>3</td>
<td>8</td>
<td>...</td>
<td>503</td>
<td>...</td>
<td>1508</td>
<td>...</td>
</tr>
<tr>
<td>F4</td>
<td>4</td>
<td>9</td>
<td>...</td>
<td>504</td>
<td>...</td>
<td>1509</td>
<td>...</td>
</tr>
<tr>
<td>F5</td>
<td>5</td>
<td>10</td>
<td>...</td>
<td>505</td>
<td>...</td>
<td>1510</td>
<td>...</td>
</tr>
</tbody>
</table>

The multiples of 503 least than 2515 that are the product of 5 by 503 are 503, 1006, 1509 and 2012 respectively (marked in bold in the array). To known in which row and column lie theses numbers it is easy to observe that, for instance, 503 = 100 \times 5 + 3 will be in row \(F3\) and column \(Cx\) where \(x\) is the solution of the equation \(5x – 2 = 503\). That is, in the column \(C101\).
Likewise, we get the rows and columns where lie the other multiples of 503. That is, $C_{202}, C_{302}$ and $C_{403}$.

Once the array is build up it is easy to realize that all the numbers in the last row belong to $A$ and all the numbers of the preceding rows bigger than the ones marked in bold belong to $A$ too. Since the rightest number is 2012, then the number searched is 2007. Now, will be suffice to see that 2007 does not belong to $A$. Indeed, suppose that there exist two nonnegative integers $a$ and $b$ such that $5a + 503b = 2007$ (Notice that $0 \leq b \leq 3$). Since $2007 = 5 \times 503 - (5 + 503)$, then $5 \times 503 = 5(a + 1) + 503(b + 1)$ and $5|503(b + 1)$. Since $\gcd(5, 503) = 1$, then $5|b + 1$ which impossible because $b + 1 < 5$. Contradiction and we are done.

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Also solved by José Gibergans-Báguena, Barcelona Tech, Barcelona, Spain.

MC–19. Let $x_1, x_2, \ldots, x_n$ be real numbers. Prove that,

$$\left( \frac{1}{n} \sum_{k=1}^{n} \cosh x_k \right)^2 \geq 1 + \left( \frac{1}{n} \sum_{k=1}^{n} \sinh x_k \right)^2$$

(IMAC Longlist – 2007)

Solution 1. The inequality is equivalent to

$$\left( \sum_{k=1}^{n} \cosh x_k \right)^2 - \left( \sum_{k=1}^{n} \sinh x_k \right)^2 \geq n^2 \quad (6)$$

The left-hand side of (6) may be written as follows:

$$\left( \sum_{k=1}^{n} \cosh x_k \right)^2 - \left( \sum_{k=1}^{n} \sinh x_k \right)^2 = \left[ \left( \sum_{k=1}^{n} \cosh x_k \right) - \left( \sum_{k=1}^{n} \sinh x_k \right) \right] \left[ \left( \sum_{k=1}^{n} \cosh x_k \right) + \left( \sum_{k=1}^{n} \sinh x_k \right) \right]$$

$$= \left[ \sum_{k=1}^{n} (\cosh x_k - \sinh x_k) \right] \left[ \sum_{k=1}^{n} (\cosh x_k + \sinh x_k) \right]$$
Solution 2. First, we write the inequality claimed in the most convenient form

\[
\left( \sum_{k=1}^{n} \cosh x_k \right)^2 - \left( \sum_{k=1}^{n} \sinh x_k \right)^2 \geq n^2
\]

and we argue by mathematical induction. The statement obviously holds when \( n = 1 \). For \( n = 2 \), we also have

\[
(cosh x_1 + cosh x_2)^2 - (sinh x_1 + sinh x_2)^2 = 2 + 2(cosh x_1 cosh x_2 - sinh x_1 sinh x_2) = 2 + 2cosh(x_1 - x_2) \geq 4
\]

Assume that the inequality holds for \( n - 1 \). We should prove

\[
\left( \sum_{k=1}^{n} \cosh x_k \right)^2 - \left( \sum_{k=1}^{n} \sinh x_k \right)^2 \geq n^2.
\]

Indeed, the LHS of the preceding inequality can be written as

\[
\left( \sum_{k=1}^{n-1} \cosh x_k + \cosh x_n \right)^2 - \left( \sum_{k=1}^{n-1} \sinh x_k + \sinh x_n \right)^2
\]

\[
= \left( \sum_{k=1}^{n-1} \cosh x_k \right)^2 + 2 \cosh x_n \sum_{k=1}^{n-1} \cosh x_k + \cosh^2 x_n
\]

\[
- \left( \sum_{k=1}^{n-1} \sinh x_k \right)^2 - 2 \sinh x_n \sum_{k=1}^{n-1} \sinh x_k - \sinh^2 x_n
\]

\[
\geq (n - 1)^2 + 1 + 2 \sum_{k=1}^{n-1} (cosh x_k cosh x_n - sinh x_k sinh x_n)
\]

\[
= (n - 1)^2 + 1 + 2 \sum_{k=1}^{n-1} \cosh(x_n - x_k) \geq n^2,
\]

and we are done.
MC–20. Let \( a, b, c \) be distinct nonzero complex numbers. Prove that
\[
\frac{a^2(1 + b^2c^2)}{(a - b)(a - c)} + \frac{b^2(1 + a^2c^2)}{(b - a)(b - c)} + \frac{c^2(1 + a^2b^2)}{(c - a)(c - b)}
\]
is an integer and determine it.

(Spanish Training Team for IMC – 2002)

**Solution 1.** The proposed expression, say \( S \), may be written as
\[
S = S_1 + S_2,
\]
where
\[
S_1 = \frac{a^2}{(a - b)(a - c)} + \frac{b^2}{(b - a)(b - c)} + \frac{c^2}{(c - a)(c - b)},
\]
\[
S_2 = a^2b^2c^2 \left( \frac{1}{(a - b)(a - c)} + \frac{1}{(b - a)(b - c)} + \frac{1}{(c - a)(c - b)} \right).
\]
Note that, since \((a - b) + (b - c) + (c - a) = 0\), then \( S_2 = 0 \). We shall show that \( S_1 = 1 \) from where we conclude that \( S = S_1 = 1 \):
\[
S_1 = \frac{a^2(b - c) - b^2(a - c) + c^2(a - b)}{(a - b)(a - c)(b - c)}
\]
\[
= \frac{a^2(b - c) - b^2(a - b) - b^2(b - c) + c^2(a - b)}{(a - b)(a - c)(b - c)}
\]
\[
= \frac{(a^2 - b^2)(b - c) + (a - b)(c^2 - b^2)}{(a - b)(a - c)(b - c)}
\]
\[
= \frac{(a - b)(b - c)(a + b - c - b)}{(a - b)(a - c)(b - c)} = 1.
\]

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**Solution 2.** We claim that the rational expression in the statement is equal to 1. Indeed, it is well known from the theory of divided differences [1] that
\[
f(z_0, z_1, \cdots , z_n) = \sum_{j=0}^{n} f(z_j) \prod_{k=0 \atop k \neq j}^{n} \frac{1}{z_j - z_k}
\]
Applying the preceding result to the function \( f(z) = z^2 + a^2b^2c^2 \), the LHS of the above identity equals to

\[
f(a, b, c) = \frac{f(b, c) - f(a, b)}{c - a} = \frac{1}{c - a} \left[ (b + c) - (a + b) \right] = 1
\]

and the RHS is

\[
a^2(1 + b^2c^2) + \frac{b^2(1 + a^2c^2)}{(b - a)(c - a)} + \frac{c^2(1 + a^2b^2)}{(c - a)(c - b)}
\]

and we are done.


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