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New inequalities for the triangle

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ABSTRACT. In this paper we will prove some new inequalities for the triangle. Among these, we will improve Euler's Inequality, Mitrinović's Inequality and Weitzenböck's Inequality, thus:

$$R \geq \frac{4}{\sum\limits_{cyclic} \sqrt{F_\lambda\left(\frac{1}{h_a}, \frac{1}{h_b}\right)(n)}} \geq 2r; \; s \geq \frac{1}{2} \sum_{cyclic} \sqrt{F_\lambda (s-a, s-b)(n)} \geq 3\sqrt{3}r;$$

and

$$a^{2\alpha} + b^{2\alpha} + c^{2\alpha} \geq \frac{1}{2} \sum_{cyclic} \sqrt{F_\lambda \left(a^{2\alpha}, b^{2\alpha}\right)(n)} \geq 3\left(\frac{4\Delta}{\sqrt{3}}\right)^\alpha,$$

where $F_\lambda(x,y)(n) = [(1+(1-2\lambda)^n)x + (1-(1-2\lambda)^n)y] \cdot$
$\cdot [(1-(1-2\lambda)^n) + (1+(1-2\lambda)^n)y]$, with $\lambda \in [0,1]$, for any $x, y \geq 0$ and for all integers $n \geq 0$.

1. INTRODUCTION

Among well known the geometric inequalities, we recall the famous inequality of Euler, $R \geq 2r$, the inquality of Mitrinović, $s \geq 3\sqrt{3}r$, and in the year 1919 Weitzenböck published in Mathematische Zeitschrift the following inequality,

$$a^2 + b^2 + c^2 \geq 4\sqrt{3}\Delta.$$

This inequality later, in 1961, was given at the International Mathematical Olympiad. In 1927, this inequality appeared as the generalization

$$\Delta \leq \frac{\sqrt{3}}{4}\left(\frac{a^k + b^k + c^k}{3}\right)^{\frac{2}{k}},$$

in one of the issues of the *American Mathematical Monthly*. For $k = 2$, we obtain the Weitzenböck Inequality.
In this paper we will prove several improvements for these inqualities.

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2. MAIN RESULTS

In the following, we will use the notations: $a, b, c-$ the lengths of the sides, h_a, h_b, h_c- the lengths of the altitudes, r_a, r_b, r_c- the radii of the excircles, s is the semi-perimeter; R is the circumradius, $r-$ the inradius, and $\Delta-$ the area of the triangle ABC.

Lemma 2.1 If $x, y \geq 0$ and $\lambda \in [0,1]$, then the inequality

$$\left(\frac{x+y}{2}\right)^2 \geq [(1-\lambda)\,x+\lambda y]\cdot[\lambda x+(1-\lambda)\,y] \geq xy \tag{2.1}$$

holds.
Proof. The inequality

$$\left(\frac{x+y}{2}\right)^2 \geq [(1-\lambda)\,x+\lambda y]\cdot[\lambda x+(1-\lambda)\,y]$$

is equivalent to

$$(1-2\lambda)^2\,x^2-2\,(1-2\lambda)^2\,xy+(1-2\lambda)^2\,y^2 \geq 0,$$

which means that

$$(1-2\lambda)^2\,(x-y)^2 \geq 0,$$

which is true. The equality holds if and only if $\lambda=\frac{1}{2}$ or $x=y$.

The inequality

$$[(1-\lambda)\,x+\lambda y]\cdot[\lambda x+(1-\lambda)\,y] \geq xy$$

becomes

$$\lambda\,(1-\lambda)\,x^2-2\lambda\,(1-\lambda)\,xy+\lambda\,(1-\lambda)\,y^2 \geq 0$$

Therefore, we obtain

$$\lambda\,(1-\lambda)\,(x-y)^2 \geq 0,$$

which is true, because $\lambda \in [0,1]$. The equality holds if and only if $\lambda \in \{0,1\}$ or $x=y$.
We consider the expression

$$F_\lambda\,(x,y)\,(n)=[(1+(1-2\lambda)^n)\,x+(1-(1-2\lambda)^n)\,y]\cdot$$

$$\cdot\left[\left(1-(1-2\lambda)^{n}\right)x+\left(1+(1-2\lambda)^{n}\right)y\right],$$

with $\lambda\in[0,1]$, for any $x,y\geq 0$, and for all integers $n\geq 0$.

Theorem 2.2 There are the following relations:

$$F_{\lambda}\left((1-\lambda)x+\lambda y,\lambda x+(1-\lambda)y\right)(n)=F_{\lambda}(x,y)(n+1); \tag{2.2}$$

$$F_{\lambda}(x,y)(n+1)\geq F_{\lambda}(x,y)(n) \tag{2.3}$$

and

$$(x+y)^{2}\geq F_{\lambda}(x,y)(n)\geq 4xy, \tag{2.4}$$

for any $\lambda\in[0,1]$,for any $x,y\geq 0$ and all integers $n\geq 0$.
Proof. We make the following calculation:

$$F_{\lambda}\left((1-\lambda)x+\lambda y,\lambda x+(1-\lambda)y\right)(n)=$$

$$=\left[\left(1+(1-2\lambda)^{n}\right)\left((1-\lambda)x+\lambda y\right)+\left(1-(1-2\lambda)^{n}\right)\left(\lambda x+(1-\lambda)y\right)\right]\cdot$$

$$\cdot\left[\left(1-(1-2\lambda)^{n}\right)\left((1-\lambda)x+\lambda y\right)+\left(1+(1-2\lambda)^{n}\right)\left(\lambda x+(1-\lambda)y\right)\right]=$$

$$=\{\left[1-\lambda+(1-\lambda)(1-2\lambda)^{n}+\lambda-\lambda(1-2\lambda)^{n}\right]x+$$

$$+\left[\lambda+\lambda(1-2\lambda)^{n}+1-\lambda-(1-\lambda)(1-2\lambda)^{n}\right]y\}$$

$$\cdot\{\left[1-\lambda-(1-\lambda)(1-2\lambda)^{n}+\lambda+\lambda(1-2\lambda)^{n}\right]x+$$

$$+\left[\lambda-(1-2\lambda)^{n}+1-\lambda+(1-\lambda)(1-2\lambda)^{n}\right]y\}=$$

$$=\left[\left(1+(1-2\lambda)^{n+1}\right)x+\left(1-(1-2\lambda)^{n+1}\right)y\right]$$

$$\left[\left(1-(1-2\lambda)^{n+1}\right)x+\left(1+(1-2\lambda)^{n+1}\right)y\right]=$$

$$= F_\lambda(x,y)(n+1),$$

so $F_\lambda((1-\lambda)x+\lambda y, \lambda x+(1-\lambda)y)(n) = F_\lambda(x,y)(n+1)$.
We use the induction on n. For $n=0$, we obtain the inequality

$$F_\lambda(x,y)(1) \geq F_\lambda(x,y)(0).$$

Therefore, we deduce the following inequality:

$$4[(1-\lambda)x+\lambda y]\cdot[\lambda x+(1-\lambda)y] \geq 4xy,$$

which is true, from Lemma 2.1.
We assume it is true for every integer $\leq n$, so

$$F_\lambda(x,y)(n+1) \geq F_\lambda(x,y)(n)$$

We will prove that

$$F_\lambda(x,y)(n+2) \geq F_\lambda(x,y)(n+1). \tag{2.5}$$

Using the substitutions $x \to (1-\lambda)x+\lambda y$ and $y \to \lambda x+(1-\lambda)y$ in the inequality (2.3), we deduce

$$F_\lambda((1-\lambda)x+\lambda y, \lambda x+(1-\lambda)y)(n+1) \geq$$

$$\geq F_\lambda((1-\lambda)x+\lambda y, \lambda x+(1-\lambda)y)(n),$$

so, from equality (2.2), we have

$$F_\lambda(x,y)(n+2) \geq F_\lambda(x,y)(n+1).$$

so we obtain (2.6).
According to inequality (2.3), we can write the sequence of inequalities

$$F_\lambda(x,y)(n) \geq F_\lambda(x,y)(n-1) \geq ... \geq F_\lambda(x,y)(1) \geq F_\lambda(x,y)(0) = 4xy.$$

Therefore, we have

$$F_\lambda(x,y)(n) \geq 4xy, \text{ for any } \lambda \in [0,1], x,y \geq 0 \text{ and for all integers } n \geq 0.$$

If $\lambda \in (0,1)$, then $1-2\lambda \in (-1,1)$ and passing to limit when $n \to \infty$, we obtain

$$\lim_{n\to\infty} F_\lambda(x,y)(n) = (x+y)^2.$$

Since the sequence $(F_\lambda(x,y)(n))_{n\geq 0}$ is increasing, we deduce

$$(x+y)^2 \geq F_\lambda(x,y)(n), \text{ for any } \lambda \in (0,1), x, y \geq 0 \text{ and for all integers } n \geq 0.$$

From the inequalities above we have that

$$(x+y)^2 \geq F_\lambda(x,y)(n) \geq 4xy, \text{ for any } \lambda \in (0,1), x, y \geq 0 \text{ and for all integers } n \geq 0.$$

If $\lambda = 0$ and $\lambda = 1$, then $F_\lambda(x,y)(n) = 4xy$, so

$$(x+y)^2 \geq F_\lambda(x,y)(n) \geq 4xy.$$

It follows that

$$(x+y)^2 \geq F_\lambda(x,y)(n) \geq 4xy, \text{ for any } \lambda \in [0,1], x, y \geq 0 \text{ and for all integers } n \geq 0.$$

Thus, the proof of Theorem 2.2 is complete.

Remark 1. *It is easy to see that there is the sequence of inequalities*

$$(x+y)^2 \geq ... \geq F_\lambda(x,y)(n) \geq F_\lambda(x,y)(n-1) \geq ...$$

$$... \geq F_\lambda(x,y)(1) \geq F_\lambda(x,y)(0) = 4xy. \tag{2.6}$$

Corollary 2.3. There are the following inequalities:

$$x+y \geq \sqrt{F_\lambda(x,y)(n)} \geq 2\sqrt{xy}; \tag{2.7}$$

$$x^2+y^2 \geq \sqrt{F_\lambda(x^2,y^2)(n)} \geq 2xy; \tag{2.8}$$

$$x+y+z \geq \frac{1}{2}\sum_{cyclic}\sqrt{F_\lambda(x,y)(n)} \geq \sqrt{xy}+\sqrt{yz}+\sqrt{zx}; \tag{2.9}$$

$$x^2+y^2+z^2 \geq \frac{1}{2}\sum_{cyclic}\sqrt{F_\lambda\left(x^2,y^2\right)(n)} \geq xy+yz+zx; \tag{2.10}$$

$$x^2+y^2+z^2+xy+yz+zx \geq \frac{1}{2}\sum_{cyclic}F_\lambda\left(x,y\right)(n) \geq 2\left(xy+yz+zx\right) \tag{2.11}$$

and

$$\left(x+y\right)\left(y+z\right)\left(z+x\right) \geq \sqrt{\prod_{cyclic}F_\lambda\left(x,y\right)(n)} \geq 8xyz, \tag{2.12}$$

for any $\lambda \in [0,1]$, for any $x, y \geq 0$, and for all integers $n \geq 0$.
Proof. From Theorem 2.2, we easily deduce inequality (2.7). Using the substitutions $x \to x^2$ and $y \to y^2$ in inequality (2.7), we obtain inequality (2.8). Similarly to inequality (2.7), $x+y \geq \sqrt{F_\lambda\left(x,y\right)(n)} \geq 2\sqrt{xy}$, we can write the following inequalities:

$$y+z \geq \sqrt{F_\lambda\left(y,z\right)(n)} \geq 2\sqrt{yz} \text{ and } z+x \geq \sqrt{F_\lambda\left(z,x\right)(n)} \geq 2\sqrt{zx},$$

which means, by adding, that

$$x+y+z \geq \frac{1}{2}\sum_{cyclic}\sqrt{F_\lambda\left(x,y\right)(n)} \geq \sqrt{xy}+\sqrt{yz}+\sqrt{zx}.$$

It is easy to see that, by making the substitutions $x \to x^2$ and $y \to y^2$ in inequality (2.9), we obtain inequality (2.10). Similar to inequality (2.4), $\left(x+y\right)^2 \geq F_\lambda\left(x,y\right)(n) \geq 4xy$, we obtain the following inequalities:

$$\left(y+z\right)^2 \geq F_\lambda\left(y,z\right)(n) \geq 4yz \text{ and } \left(z+x\right)^2 \geq F_\lambda\left(z,x\right)(n) \geq 4zx.$$

By adding them, we have inequality (2.11) and by multiplying them, we obtain inequality (2.12).

Lemma 2.4 For any triangle ABC, the following inequality,

$$\sqrt{ab}+\sqrt{bc}+\sqrt{ca} \geq \frac{4\Delta}{R}, \tag{2.13}$$

holds.
Proof. We apply the arithmetic-geometric mean inequality and we find that

$$\sqrt{ab}+\sqrt{bc}+\sqrt{ca} \geq 3\sqrt[3]{abc}.$$

It is sufficient to show that

$$3\sqrt[3]{abc} \geq \frac{4\Delta}{R}. \tag{2.14}$$

Inequality (2.14) is equivalent to

$$27abc \geq \frac{64\Delta^3}{R^3},$$

so

$$27 \cdot 4R\Delta \geq \frac{64\Delta^3}{R^3},$$

which means that

$$27R^4 \geq 16\Delta^2. \tag{2.15}$$

Using Mitrinović's Inequality, $3\sqrt{3}R \geq 2s$, and Euler's Inequality $R \geq 2r$, we deduce, by multiplication, that

$$3\sqrt{3}R^2 \geq 4\Delta.$$

It follows (2.15) .

Corollary 2.5. In any triangle ABC, there are the following inequalities:

$$R \geq \frac{4}{\sum\limits_{cyclic} \sqrt{F_\lambda\left(\frac{1}{h_a}, \frac{1}{h_b}\right)(n)}} \geq 2r; \tag{2.16}$$

$$s \geq \frac{1}{2} \sum_{cyclic} \sqrt{F_\lambda\left(s-a, s-b\right)(n)} \geq 3\sqrt{3}r \tag{2.17}$$

and

$$a^{2\alpha} + b^{2\alpha} + c^{2\alpha} \geq \frac{1}{2} \sum_{cyclic} \sqrt{F_\lambda\left(a^{2\alpha}, b^{2\alpha}\right)(n)} \geq 3\left(\frac{4\Delta}{\sqrt{3}}\right)^\alpha, \tag{2.18}$$

for any $\lambda \in [0,1]$, $x, y \geq 0$, $n \geq 0$ and α is a real numbers.

Proof. Making the substitutions $x = \frac{1}{h_a}, y = \frac{1}{h_b}$ and $z = \frac{1}{h_c}$ in inequality (2.9), we obtain

$$\frac{1}{h_a}+\frac{1}{h_b}+\frac{1}{h_c} \geq \frac{1}{2} \sum_{cyclic} \sqrt{F_\lambda\left(\frac{1}{h_a}, \frac{1}{h_b}\right)(n)} \geq \frac{1}{\sqrt{h_a h_b}}+\frac{1}{\sqrt{h_b h_c}}+\frac{1}{\sqrt{h_c h_a}}. \tag{2.19}$$

According to the equalities

$$h_a = \frac{2\Delta}{a}, h_b = \frac{2\Delta}{b} \text{ and } h_c = \frac{2\Delta}{c},$$

we have

$$\frac{1}{\sqrt{h_a h_b}} + \frac{1}{\sqrt{h_b h_c}} + \frac{1}{\sqrt{h_c h_a}} = \frac{1}{2\Delta}\left(\sqrt{ab} + \sqrt{bc} + \sqrt{ca}\right).$$

From Lemma 2.4, we deduce

$$\frac{1}{\sqrt{h_a h_b}} + \frac{1}{\sqrt{h_b h_c}} + \frac{1}{\sqrt{h_c h_a}} \geq \frac{2}{R}.$$

If we use the identity

$$\frac{1}{h_a} + \frac{1}{h_b} + \frac{1}{h_c} = \frac{1}{r}$$

and inequality from above then inequality (2.18) becomes

$$\frac{1}{r} \geq \frac{1}{2} \sum_{cyclic} \sqrt{F_\lambda \left(\frac{1}{h_a}, \frac{1}{h_b}\right)(n)} \geq \frac{2}{R}. \tag{2.20}$$

Consequently the inequalities (2.16) follows.
If in inequality (2.9) we take $x = s - a, y = s - b$ and $z = s - c$, then we deduce the inequality

$$s \geq \frac{1}{2} \sum_{cyclic} \sqrt{F_\lambda (s-a, s-b)(n)} \geq$$

$$\geq \sqrt{(s-a)(s-b)} + \sqrt{(s-b)(s-c)} + \sqrt{(s-c)(s-a)}. \tag{2.21}$$

But, we know the identity $\sum\limits_{cyclic} \sqrt{(s-a, s-b)} = \sum\limits_{cyclic} \sqrt{bc} \sin \frac{A}{2}$.
Using the arithmetic-geometric mean inequality, we obtain

$$\begin{aligned} \sum_{cyclic} \sqrt{bc} \sin \frac{A}{2} &\geq 3\sqrt[3]{abc \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}} = 3\sqrt[3]{4R\Delta \cdot \frac{r}{4R}} = \\ = 3\sqrt[3]{\Delta r} &= 3\sqrt[3]{sr^2} \geq 3\sqrt[3]{3\sqrt{3} r^3} = 3\sqrt{3} r. \end{aligned}$$

Hence,

$$\sqrt{(s-a)(s-b)}+\sqrt{(s-b)(s-c)}+\sqrt{(s-c)(s-a)}\geq 3\sqrt{3}r, \qquad (2.22)$$

which means, according to inequalities (2.21) and (2.22), that

$$s\geq \frac{1}{2}\sum_{cyclic}\sqrt{F_{\lambda}(s-a,s-b)(n)}\geq 3\sqrt{3}r.$$

Making the substitutions $x=a^{\alpha}, y=b^{\alpha}$, and $z=c^{\alpha}$ in inequality (2.9), we obtain the following inequality:

$$a^{2\alpha}+b^{2\alpha}+c^{2\alpha}\geq \frac{1}{2}\sum_{cyclic}\sqrt{F_{\lambda}(a^{2\alpha},b^{2\alpha})(n)}\geq a^{\alpha}b^{\alpha}+b^{\alpha}c^{\alpha}+c^{\alpha}a^{\alpha}. \qquad (2.23)$$

Applying the arithmetic- geometric mean inequality and Pólya-Szegő's Inequality, $\sqrt[3]{a^2b^2c^2}\geq \frac{4\Delta}{\sqrt{3}}$, we deduce

$$a^{\alpha}b^{\alpha}+b^{\alpha}c^{\alpha}+c^{\alpha}a^{\alpha}\geq \sqrt[3]{(a^2b^2c^2)^{\alpha}}=3\left(\sqrt[3]{a^2b^2c^2}\right)^{\alpha}\geq 3\left(\frac{4\Delta}{\sqrt{3}}\right)^{\alpha},$$

so

$$a^{\alpha}b^{\alpha}+b^{\alpha}c^{\alpha}+c^{\alpha}a^{\alpha}\geq 3\left(\frac{4\Delta}{\sqrt{3}}\right)^{\alpha}. \qquad (2.24)$$

According to inequalities (2.23) and (2.24), we obtain the inequality

$$a^{2\alpha}+b^{2\alpha}+c^{2\alpha}\geq \frac{1}{2}\sum_{cyclic}\sqrt{F_{\lambda}(a^{2\alpha},b^{2\alpha})(n)}\geq 3\left(\frac{4\Delta}{\sqrt{3}}\right)^{\alpha}.$$

Thus, the statement is true.

Remark 2. *a) Inequality (2.16) implies the sequence of inequalities*

$$R\geq \frac{4}{\sum\limits_{cyclic}\sqrt{F_{\lambda}\left(\frac{1}{h_a},\frac{1}{h_b}\right)(0)}}\geq ...\geq \frac{4}{\sum\limits_{cyclic}\sqrt{F_{\lambda}\left(\frac{1}{h_a},\frac{1}{h_b}\right)(n-1)}}\geq$$

$$\geq \frac{4}{\sum\limits_{cyclic}\sqrt{F_{\lambda}\left(\frac{1}{h_a},\frac{1}{h_b}\right)(n)}}\geq ...\geq 2r \qquad (2.25)$$

b) For $\alpha=1$ in inequality (2.18), we obtain

$$a^2+b^2+c^2 \geq \frac{1}{2}\sum_{cyclic}\sqrt{F_\lambda\left(a^2,b^2\right)(n)} \geq 4\sqrt{3}\Delta, \tag{2.26}$$

which proves Weitzenböck's Inequality, namely

$$a^2+b^2+c^2 \geq 4\sqrt{3}\Delta.$$

Corollary 2.6. For any triangle ABC, there are the following inequalities:

$$2\left(s^2-r^2-4Rr\right) \geq \frac{1}{2}\sum_{cyclic}\sqrt{F_\lambda\left(a^2,b^2\right)(n)} \geq s^2+r^2+4Rr, \tag{2.27}$$

$$s^2-2r^2-8Rr \geq \frac{1}{2}\sum_{cyclic}\sqrt{F_\lambda\left((s-a)^2,(s-b)^2\right)(n)} \geq r\left(4R+r\right), \tag{2.28}$$

$$\frac{\left(s^2+r^2+4Rr\right)^2-8s^2Rr}{4R^2} \geq \frac{1}{2}\sum_{cyclic}\sqrt{F_\lambda\left(h_a^2,h_b^2\right)(n)} \geq \frac{2s^2r}{R}, \tag{2.29}$$

$$(4R+r)^2-2s^2 \geq \frac{1}{2}\sum_{cyclic}\sqrt{F_\lambda\left(r_a^2,r_b^2\right)(n)} \geq s^2, \tag{2.30}$$

$$\frac{8R^2+r^2-s^2}{8R^2} \geq \frac{1}{2}\sum_{cyclic}\sqrt{F_\lambda\left(\sin^4\frac{A}{2},\sin^4\frac{B}{2}\right)(n)} \geq \frac{s^2+r^2-8Rr}{16R^2} \tag{2.31}$$

and

$$\frac{(4R+r)^2-s^2}{4R^2} \geq \frac{1}{2}\sum_{cyclic}\sqrt{F_\lambda\left(\cos^4\frac{A}{2},\cos^4\frac{B}{2}\right)(n)} \geq$$

$$\geq \frac{s^2+(4R+r)^2}{8R^2}. \tag{2.32}$$

Proof. According to Corollary 2.3 we have the inequality

$$x^2+y^2+z^2 \geq \frac{1}{2}\sum_{cyclic}\sqrt{F_\lambda\left(x^2,y^2\right)(n)} \geq xy+yz+zx.$$

Using the substitutions

$$\begin{aligned}(x,y,z) \quad \in \quad & \{(a,b,c),(s-a,s-b,s-c),(h_a,h_b,h_c),(r_a,r_b,r_c), \\ & \left(\sin^2\frac{A}{2},\sin^2\frac{B}{2},\sin^2\frac{C}{2}\right),\left(\cos^2\frac{A}{2},\cos^2\frac{B}{2},\cos^2\frac{C}{2}\right)\Bigg\},\end{aligned}$$

we deduce the inequalities required.

Corollary 2.7. In any triangle ABC, there are the following inequalities:

$$3s^2-r^2-4Rr \geq \frac{1}{2}\sum_{cyclic} F_\lambda(a,b)(n) \geq 2\left(s^2+r^2+4Rr\right), \tag{2.33}$$

$$s^2-r^2-4Rr \geq \frac{1}{2}\sum_{cyclic} F_\lambda(s-a,s-b)(n) \geq 2r(4R+r), \tag{2.34}$$

$$\frac{\left(s^2+r^2+4Rr\right)^2-8s^2Rr}{4R^2} \geq \frac{1}{2}\sum_{cyclic} F_\lambda(h_a,h_b)(n) \geq \frac{4s^2r}{R} \tag{2.35}$$

and

$$(4R+r)^2-s^2 \geq \frac{1}{2}\sum_{cyclic} F_\lambda(r_a,r_b)(n) \geq 2s^2 \tag{2.36}$$

Proof. According to Corollary 2.3, we have the inequality

$$x^2+y^2+z^2+xy+yz+zx \geq \frac{1}{2}\sum_{cyclic} F_\lambda(x,y)(n) \geq 2(xy+yz+zx).$$

Using the substitutions

$$(x,y,z) \in \{(a,b,c),(s-a,s-b,s-c),(h_a,h_b,h_c),(r_a,r_b,r_c)\}$$

we deduce the inequalities from the statement.

Corollary 2.8. For any triangle ABC there are the following inequalities:

$$2s\left(s^2+r^2+2Rr\right) \geq \prod_{cuclic}\sqrt{F_\lambda(a,b)(n)} \geq 32sRr, \tag{2.37}$$

$$4sRr \geq \prod_{cyclic}\sqrt{F_\lambda((s-a),(s-b))(n)} \geq 8sr^2, \tag{2.38}$$

$$\frac{s^2r\left(s^2+r^2+4Rr\right)}{R^2} \geq \prod_{cyclic} \sqrt{F_\lambda\left(h_a,h_b\right)(n)} \geq \frac{16s^2r^2}{R}, \tag{2.39}$$

$$4s^2R \geq \prod_{cyclic} \sqrt{F_\lambda\left(r_a,r_b\right)(n)} \geq 8s^2r, \tag{2.40}$$

$$\frac{(2R-r)\left(s^2+r^2-8Rr\right)-2Rr^2}{32R^3} \geq \prod_{cyclic} \sqrt{F_\lambda\left(\sin^2\frac{A}{2},\sin^2\frac{B}{2}\right)(n)} \geq$$

$$\geq \frac{r^2}{2R^2} \tag{2.41}$$

and

$$\frac{(4R+r)^3+s^2(2R+r)}{32R^3} \geq \prod_{cyclic} \sqrt{F_\lambda\left(\cos^2\frac{A}{2},\cos^2\frac{B}{2}\right)(n)} \geq \frac{s^2}{2R^2} \tag{2.42}$$

Proof. According to Corollary 2.3, we have the inequality

$$(x+y)(y+z)(z+x) \geq \sqrt{\prod_{cyclic} F_\lambda(x,y)(n)} \geq 8xyz.$$

Using the substitutions

$$\begin{aligned}(x,y,z) \quad \in \quad & \{(a,b,c),(s-a,s-b,s-c),(h_a,h_b,h_c),(r_a,r_b,r_c), \\ & \left(\sin^2\frac{A}{2},\sin^2\frac{B}{2},\sin^2\frac{C}{2}\right),\left(\cos^2\frac{A}{2},\cos^2\frac{B}{2},\cos^2\frac{C}{2}\right)\right\},\end{aligned}$$

we deduce the inequalities required.

Remark 3. *From Corollary 2.7, we obtain the inequality*

$$\begin{aligned}2s\left(s^2+r^2+2Rr\right) \quad & \geq \quad \prod_{cyclic} \sqrt{F_\lambda(a,b)(n)} \geq 32sRr \geq \\ & \geq \quad 8\prod_{cyclic} \sqrt{F_\lambda((s-a),(s-b))(n)} \geq 64sr^2.\end{aligned}$$

We consider the expression

$$G(x,y)(n) = \frac{xy\left(x^{n-1}+y^{n-1}\right)\left(x^{n+1}+y^{n+1}\right)}{\left(x^{n}+y^{n}\right)^{2}}, \tag{2.43}$$

where $x, y > 0$ and for all integers $n \geq 0$.

Theorem 2.9. For any $x, y > 0$ and for all integers $n \geq 0$, there are the following relations:

$$a)\ \left(\frac{x+y}{2}\right)^{2} \geq G(x,y)(n) \geq xy \tag{2.44}$$

and

$$b)\ G(x,y)(n+1) \leq G(x,y)(n). \tag{2.45}$$

Proof. We take $\lambda = \frac{x^n}{x^n+y^n}$,for all integers $n \geq 0$, in inequality (2.1), because $\lambda \in (0,1)$, and we deduce

$$\left(\frac{x+y}{2}\right)^{2} \geq \frac{xy\left(x^{n-1}+y^{n-1}\right)\left(x^{n+1}+y^{n+1}\right)}{\left(x^{n}+y^{n}\right)^{2}} \geq xy,$$

so,

$$\left(\frac{x+y}{2}\right)^{2} \geq G(x,y)(n) \geq xy.$$

To prove inequality (2.45), we can write

$$\frac{G(x,y)(n+1)}{G(x,y)(n)} - 1 =$$

$$= -\frac{(xy)^{n-1}(x-y)^{2}\left(x^{2}+xy+y^{2}\right)\left(x^{2n}+x^{2n-1}y+x^{2n}y^{2}+...+y^{2n}\right)}{\left(x^{n+1}+y^{n+1}\right)^{3}\left(x^{n-1}+y^{n-1}\right)} \leq 0.$$

Consequently, we have

$$G(x,y)(n+1) \leq G(x,y)(n).$$

Remark 4. *It is easy to see that there is the sequence of inequalities*

$$\frac{(x+y)^{2}}{4} = G(x,y)(0) \geq G(x,y)(1) \geq ...$$

$$\geq G(x,y)(n-1) \geq G(x,y)(n) \geq ... \geq xy. \tag{2.46}$$

Corollary 2.10
There are the following inequalities:

$$\frac{x+y}{2} \geq \sqrt{G(x,y)(n)} \geq \sqrt{xy}; \tag{2.47}$$

$$\frac{x^2+y^2}{2} \geq \sqrt{G(x^2,y^2)(n)} \geq \sqrt{xy}; \tag{2.48}$$

$$x+y+z \geq \sum_{cyclic} \sqrt{G(x,y)(n)} \geq \sqrt{xy}+\sqrt{yz}+\sqrt{zx}; \tag{2.49}$$

$$x^2+y^2+z^2 \geq \sum_{cyclic} \sqrt{G(x^2,y^2)(n)} \geq xy+yz+zx; \tag{2.50}$$

$$\frac{1}{2}\left(x^2+y^2+z^2+xy+yz+zx\right) \geq \sum_{cyclic} G(x,y)(n) \geq xy+yz+zx \tag{2.51}$$

and

$$\frac{1}{8}(x+y)(y+z)(z+x) \geq \sqrt{\prod_{cyclic} G(x,y)(n)} \geq xyz, \tag{2.52}$$

for any $x, y > 0$ and for all integers $n \geq 0$.
Proof. From Theorem 2.9, we easily deduce inequality (2.47). Using the substitutions $x \to x^2$ and $y \to y^2$ in inequality (2.47), we obtain inequality (2.48). Similarly to inequality (2.47), $\frac{x+y}{2} \geq \sqrt{G(x,y)(n)} \geq \sqrt{xy}$, we can write the following inequalities:

$$\frac{y+z}{2} \geq \sqrt{G(y,z)(n)} \geq \sqrt{yz} \text{ and } \frac{z+x}{2} \geq \sqrt{G(z,x)(n)} \geq \sqrt{zx},$$

which means, by adding, that

$$x+y+z \geq \sum_{cyclic} \sqrt{G(x,y)(n)} \geq \sqrt{xy}+\sqrt{yz}+\sqrt{zx}.$$

It is easy to see that by making the substitutions $x \to x^2$ and $y \to y^2$ in inequality (2.49), we obtain inequality (2.50). Similarly to inequality (2.44), $\left(\frac{x+y}{2}\right)^2 \geq G(x,y)(n) \geq xy$, we obtain the following inequalities:

$$\left(\frac{y+z}{2}\right)^2 \geq G(y,z)(n) \geq yz \text{ and } \left(\frac{z+x}{2}\right)^2 \geq G(z,x)(n) \geq zx.$$

By adding them, we have inequality (2.51) and by multiplying them, we obtain inequality (2.52).

Corollary 2.11. In any triangle ABC , there are the following inequalities:

$$R \geq \frac{2}{\sum_{cyclic} \sqrt{G\left(\frac{1}{h_a}, \frac{1}{h_b}\right)(n)}} \geq 2r; \tag{2.53}$$

$$s \geq \sum_{cyclic} \sqrt{G(s-a, s-b)(n)} \geq 3\sqrt{3}r \tag{2.54}$$

and

$$a^{2\alpha} + b^{2\alpha} + c^{2\alpha} \geq \sum_{cyclic} \sqrt{G\left(a^{2\alpha}, b^{2\alpha}\right)(n)} \geq 3\left(\frac{4\Delta}{\sqrt{3}}\right)^{\alpha}, \tag{2.55}$$

for any $n \geq 0$ and for every real numbers α.
Proof. Making the substitutions $x = \frac{1}{h_a}, y = \frac{1}{h_b}$ and $z = \frac{1}{h_c}$ in inequality (2.49), we obtain

$$\frac{1}{h_a} + \frac{1}{h_b} + \frac{1}{h_c} \geq \sum_{cyclic} \sqrt{G\left(\frac{1}{h_a}, \frac{1}{h_b}\right)(n)} \geq \frac{1}{\sqrt{h_a h_b}} + \frac{1}{\sqrt{h_b h_c}} + \frac{1}{\sqrt{h_c h_a}}. \tag{2.56}$$

From inequality (2.13), we have

$$\frac{1}{\sqrt{h_a h_b}} + \frac{1}{\sqrt{h_b h_c}} + \frac{1}{\sqrt{h_c h_a}} \geq \frac{2}{R},$$

and from the identity

$$\frac{1}{h_a} + \frac{1}{h_b} + \frac{1}{h_c} = \frac{1}{r}$$

we deduce

$$\frac{1}{r} \geq \sum_{cyclic} \sqrt{G\left(\frac{1}{h_a}, \frac{1}{h_b}\right)(n)} \geq \frac{2}{R}. \tag{2.57}$$

Consequently

$$R \geq \frac{2}{\sum\limits_{cyclic} \sqrt{G\left(\frac{1}{h_a}, \frac{1}{h_b}\right)(n)}} \geq 2r.$$

If in inequality (2.49) we take $x = s - a, y = s - b$ and $z = s - c$, then we deduce the inequality

$$s \geq \sum_{cyclic} \sqrt{G(s-a, s-b)(n)} \geq \sqrt{(s-a)(s-b)} +$$

$$+\sqrt{(s-b)(s-c)} + \sqrt{(s-c)(s-a)}. \qquad (2.58)$$

But

$$\sqrt{(s-a)(s-b)} + \sqrt{(s-b)(s-c)} + \sqrt{(s-c)(s-a)} \geq 3\sqrt{3}r,$$

which means that

$$s \geq \sum_{cyclic} \sqrt{G(s-a, s-b)(n)} \geq 3\sqrt{3}r.$$

Making the substitutions , and in inequality (2.49), we obtain the following inequality:

$$a^{2\alpha} + b^{2\alpha} + c^{2\alpha} \geq \sum_{cyclic} \sqrt{G\left(a^{2\alpha}, b^{2\alpha}\right)(n)} \geq a^\alpha b^\alpha + b^\alpha c^\alpha + c^\alpha a^\alpha. \qquad (2.59)$$

Applying the arithmetic-geometric mean inequality and Pólya-Szegö's Inequality, $\sqrt[3]{a^2b^2c^2} \geq \frac{4\Delta}{\sqrt{3}}$, we deduce

$$a^\alpha b^\alpha + b^\alpha c^\alpha + c^\alpha a^\alpha \geq \left(\frac{4\Delta}{\sqrt{3}}\right)^\alpha.$$

Therefore

$$s \geq \sum_{cyclic} \sqrt{G(s-a, s-b)(n)} \geq 3\sqrt{3}r.$$

Remark 5. *a) Inequality (2.53) implies the sequence of inequalities*

$$R \geq \ldots \frac{2}{\sum\limits_{cyclic} \sqrt{G\left(\frac{1}{h_a}, \frac{1}{h_b}\right)(n)}} \geq \frac{2}{\sum\limits_{cyclic} \sqrt{G\left(\frac{1}{h_a}, \frac{1}{h_b}\right)(n-1)}} \geq \ldots$$

$$\geq \frac{2}{\sum\limits_{cyclic} \sqrt{\left(\frac{1}{h_a}, \frac{1}{h_b}\right)(0)}} \geq 2r \tag{2.60}$$

b) For $\alpha = 1$ in inequality (2.55), we obtain

$$a^2 + b^2 + c^2 \geq \sum_{cyclic} \sqrt{G\left(a^2, b^2\right)(n)} \geq 4\sqrt{3}\Delta, \tag{2.61}$$

which proves Weitzenböck's Inequality, namely

$$a^2 + b^2 + c^2 \geq 4\sqrt{3}\Delta.$$

Corollary 2.12. For any triangle ABC , there are the following inequalities:

$$2\left(s^2 - r^2 - 4Rr\right) \geq \sum_{cyclic} \sqrt{G\left(a^2, b^2\right)(n)} \geq s^2 + r^2 + 4Rr, \tag{2.62}$$

$$s^2 - 2r^2 - 8Rr \geq \sum_{cyclic} \sqrt{G\left((s-a)^2, (s-b)^2\right)(n)} \geq r(4R + r), \tag{2.63}$$

$$\frac{\left(s^2 + r^2 + 4Rr\right)^2 - 8s^2Rr}{4R^2} \geq \sum_{cyclic} \sqrt{G\left(h_a^2, h_b^2\right)(n)} \geq \frac{2s^2r}{R}, \tag{2.64}$$

$$(4R + r)^2 - 2s^2 \geq \sum_{cyclic} \sqrt{G\left(r_a^2, r_b^2\right)(n)} \geq s^2 \tag{2.65}$$

$$\frac{8R^2 + r^2 - s^2}{8R^2} \geq \sum_{cyclic} \sqrt{G\left(\sin^4\frac{A}{2}, \sin^4\frac{B}{2}\right)(n)} \geq \frac{s^2 + r^2 - 8Rr}{16R^2} \tag{2.66}$$

and

$$\frac{4(R + r)^2 - s^2}{4R^2} \geq \sum_{cyclic} \sqrt{G\left(\cos^4\frac{A}{2}, \cos^4\frac{B}{2}\right)(n)} \geq \frac{s^2 + (4R + r)^2}{8R^2}. \tag{2.67}$$

Proof. According to Corollary 2.10, we have the inequality

$$x^2 + y^2 + z^2 \geq \sum_{cyclic} \sqrt{G\left(x^2, y^2\right)(n)} \geq xy + yz + zx.$$

Using the substitutions

$$(x,y,z) \in \{(a,b,c),(s-a,s-b,s-c),(h_a,h_b,h_c),(r_a,r_b,r_c), \left(\sin^2\frac{A}{2},\sin^2\frac{B}{2},\sin^2\frac{C}{2}\right),\left(\cos^2\frac{A}{2},\cos^2\frac{B}{2},\cos^2\frac{C}{2}\right)\right\},$$

we deduce the inequalities required.

Corollary 2.13. In any triangle ABC there are the following inequalities:

$$\frac{1}{2}\left(3s^2-r^2-4Rr\right) \geq \sum_{cyclic} G(a,b)(n) \geq s^2+r^2+4Rr, \tag{2.68}$$

$$\frac{1}{2}\left(s^2-r^2-4Rr\right) \geq \sum_{cyclic} G(s-a,s-b)(n) \geq r(4R+r), \tag{2.69}$$

$$\frac{\left(s^2+r^2+4Rr\right)^2}{8R^2} \geq \sum_{cyclic} G(h_a,h_b)(n) \geq \frac{2s^2r}{R} \tag{2.70}$$

and

$$\frac{1}{2}\left[(4R+r)^2-s^2\right] \geq \sum_{cyclic} G(r_a,r_b)(n) \geq s^2. \tag{2.71}$$

Proof. According to Corollary 2.10, we have the inequality

$$\frac{1}{2}\left(x^2+y^2+z^2+xy+yz+zx\right) \geq \sum_{cyclic} G(x,y)(n) \geq xy+yz+zx.$$

Using the substitutions

$$(x,y,z) \in \{(a,b,c),(s-a,s-b,s-c),(h_a,h_b,h_c),(r_a,r_b,r_c)\},$$

we deduce the inequalities from the statement.

Corollary 2.14. For any triangle ABC there are the following inequalities:

$$\frac{1}{4}s\left(s^2+r^2+2Rr\right) \geq \prod_{cyclic} \sqrt{G(a,b)(n)} \geq 4sRr, \tag{2.72}$$

$$\frac{1}{2}sRr \geq \prod_{cyclic} \sqrt{G((s-a),(s-b))(n)} \geq sr^2, \tag{2.73}$$

$$\frac{s^2 r\left(s^2+r^2+4Rr\right)}{8R^2} \geq \prod_{cyclic} \sqrt{G\left(h_a, h_b\right)(n)} \geq \frac{2s^2 r^2}{R}, \tag{2.74}$$

$$\frac{1}{2} s^2 R \geq \prod_{cyclic} \sqrt{G\left(r_a, r_b\right)(n)} \geq s^2 r, \tag{2.75}$$

$$\frac{(2R-r)\left(s^2+r^2-8Rr\right)-2Rr^2}{256R^3} \geq$$

$$\geq \prod_{cyclic} \sqrt{G\left(\sin^2 \frac{A}{2}, \sin^2 \frac{B}{2}\right)(n)} \geq \frac{r^2}{16R^2} \tag{2.76}$$

and

$$\frac{(4R+r)^3+s^2(2R+r)}{256R^3} \geq \prod_{cyclic} \sqrt{G\left(\cos^2 \frac{A}{2}, \cos^2 \frac{B}{2}\right)(n)} \geq \frac{s^2}{16R^2}. \tag{2.77}$$

Proof. According to Corollary 2.10, we have the inequality

$$\frac{1}{8}(x+y)(y+z)(z+x) \geq \sqrt{\prod_{cyclic} G(x,y)(n)} \geq xyz.$$

Using the substitutions

$$\begin{aligned}(x,y,z) \quad \in \quad & \{(a,b,c),(s-a,s-b,s-c),(h_a,h_b,h_c),(r_a,r_b,r_c), \\ & \left(\sin^2 \frac{A}{2}, \sin^2 \frac{B}{2}, \sin^2 \frac{C}{2}\right), \left(\cos^2 \frac{A}{2}, \cos^2 \frac{B}{2}, \cos^2 \frac{C}{2}\right)\Bigg\},\end{aligned}$$

we deduce the inequalities required.
Remark 6. From Corollary 2.13, we obtain the inequality

$$\frac{1}{4} s\left(s^2+r^2+2Rr\right) \geq \geq \prod_{cyclic} \sqrt{G(a,b)(n)} \geq 4sRr \geq$$

$$\geq 8 \prod_{cyclic} \sqrt{G((s-a),(s-b))(n)} \geq 8sr^2. \tag{2.78}$$

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