New inequalities for the triangle

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ABSTRACT. In this paper we will prove some new inequalities for the triangle. Among these, we will improve Euler’s Inequality, Mitrinović’s Inequality and Weitzenböck’s Inequality, thus:

\[ R \geq \frac{4}{\sum_{\text{cyclic}} \sqrt{F_\lambda \left( \frac{1}{n_a}, \frac{1}{n_b} \right) (n)}} \geq 2r; \quad s \geq \frac{1}{2} \sum_{\text{cyclic}} \sqrt{F_\lambda \left( s - a, s - b \right) (n)} \geq 3\sqrt{3}r; \]

and

\[ a^{2\alpha} + b^{2\alpha} + c^{2\alpha} \geq \frac{1}{2} \sum_{\text{cyclic}} \sqrt{F_\lambda \left( a^{2\alpha}, b^{2\alpha} \right) (n)} \geq 3 \left( \frac{4\Delta}{\sqrt{3}} \right)^{\alpha}, \]

where \( F_\lambda \left( x, y \right) (n) = [(1 + (1 - 2\lambda)^n) x + (1 - (1 - 2\lambda)^n) y] \cdot [(1 - (1 - 2\lambda)^n) + (1 + (1 - 2\lambda)^n) y], \) with \( \lambda \in [0, 1], \) for any \( x, y \geq 0 \) and for all integers \( n \geq 0. \)

1. INTRODUCTION

Among well known the geometric inequalities, we recall the famous inequality of Euler, \( R \geq 2r, \) the inequality of Mitrinović, \( s \geq 3\sqrt{3}r, \) and in the year 1919 Weitzenböck published in Mathematische Zeitschrift the following inequality,

\[ a^2 + b^2 + c^2 \geq 4\sqrt{3}\Delta. \]

This inequality later, in 1961, was given at the International Mathematical Olympiad. In 1927, this inequality appeared as the generalization

\[ \Delta \leq \frac{\sqrt{3}}{4} \left( \frac{a^k + b^k + c^k}{3} \right)^{\frac{2}{k}}, \]

in one of the issues of the American Mathematical Monthly. For \( k = 2, \) we obtain the Weitzenböck Inequality.

In this paper we will prove several improvements for these inequalities.

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2. MAIN RESULTS

In the following, we will use the notations: $a, b, c$—the lengths of the sides, $h_a, h_b, h_c$—the lengths of the altitudes, $r_a, r_b, r_c$—the radii of the excircles, $s$ is the semi-perimeter; $R$ is the circumradius, $r$—the inradius, and $\Delta$—the area of the triangle $ABC$.

**Lemma 2.1** If $x, y \geq 0$ and $\lambda \in [0, 1]$, then the inequality

$$\left(\frac{x + y}{2}\right)^2 \geq [(1 - \lambda) x + \lambda y] \cdot [\lambda x + (1 - \lambda) y] \geq xy \quad (2.1)$$

holds.

**Proof.** The inequality

$$\left(\frac{x + y}{2}\right)^2 \geq [(1 - \lambda) x + \lambda y] \cdot [\lambda x + (1 - \lambda) y]$$

is equivalent to

$$(1 - 2\lambda)^2 x^2 - 2(1 - 2\lambda)^2 xy + (1 - 2\lambda)^2 y^2 \geq 0,$$

which means that

$$(1 - 2\lambda)^2 (x - y)^2 \geq 0,$$

which is true. The equality holds if and only if $\lambda = \frac{1}{2}$ or $x = y$.

The inequality

$$[(1 - \lambda) x + \lambda y] \cdot [\lambda x + (1 - \lambda) y] \geq xy$$

becomes

$$\lambda (1 - \lambda) x^2 - 2\lambda (1 - \lambda) xy + \lambda (1 - \lambda) y^2 \geq 0$$

Therefore, we obtain

$$\lambda (1 - \lambda) (x - y)^2 \geq 0,$$

which is true, because $\lambda \in [0, 1]$. The equality holds if and only if $\lambda \in \{0, 1\}$ or $x = y$.

We consider the expression

$$F_\lambda (x, y) (n) = [(1 + (1 - 2\lambda)^n) x + (1 - (1 - 2\lambda)^n) y].$$
· \((1 - (1 - 2\lambda)^n) x + (1 + (1 - 2\lambda)^n) y\),

with \(\lambda \in [0, 1]\), for any \(x, y \geq 0\), and for all integers \(n \geq 0\).

**Theorem 2.2** There are the following relations:

\[
F_{\lambda} ((1 - \lambda) x + \lambda y, \lambda x + (1 - \lambda) y) (n) = F_{\lambda} (x, y) (n + 1); \quad (2.2)
\]

\[
F_{\lambda} (x, y) (n + 1) \geq F_{\lambda} (x, y) (n) \quad (2.3)
\]

and

\[
(x + y)^2 \geq F_{\lambda} (x, y) (n) \geq 4xy, \quad (2.4)
\]

for any \(\lambda \in [0, 1]\), for any \(x, y \geq 0\) and all integers \(n \geq 0\).

**Proof.** We make the following calculation:

\[
F_{\lambda} ((1 - \lambda) x + \lambda y, \lambda x + (1 - \lambda) y) (n) = \]

\[
= [(1 + (1 - 2\lambda)^n) ((1 - \lambda) x + \lambda y) + (1 - (1 - 2\lambda)^n) (\lambda x + (1 - \lambda) y)].
\]

\[
\cdot [(1 - (1 - 2\lambda)^n) ((1 - \lambda) x + \lambda y) + (1 + (1 - 2\lambda)^n) (\lambda x + (1 - \lambda) y)] =
\]

\[
= \left\{[1 - \lambda + (1 - \lambda) (1 - 2\lambda)^n + \lambda - (1 - \lambda) (1 - 2\lambda)^n] x +
\right.
\]

\[
+ [\lambda + \lambda (1 - 2\lambda)^n + 1 - \lambda - (1 - \lambda) (1 - 2\lambda)^n] y \}
\]

\[
\cdot \left\{[1 - \lambda - (1 - \lambda) (1 - 2\lambda)^n + \lambda + (1 - 2\lambda)^n] x +
\right.
\]

\[
+ [\lambda - (1 - 2\lambda)^n + 1 - \lambda + (1 - \lambda) (1 - 2\lambda)^n] y \}
\]

\[
= \left[\left(1 + (1 - 2\lambda)^{n+1}\right) x + \left(1 - (1 - 2\lambda)^{n+1}\right) y\right]
\]

\[
\left[\left(1 - (1 - 2\lambda)^{n+1}\right) x + \left(1 + (1 - 2\lambda)^{n+1}\right) y\right] =
\]
\[ F_\lambda(x, y) (n+1), \]

so \( F_\lambda((1-\lambda)x + \lambda y, \lambda x + (1-\lambda)y)(n) = F_\lambda(x, y)(n+1). \)

We use the induction on \( n \). For \( n = 0 \), we obtain the inequality

\[ F_\lambda(x, y)(1) \geq F_\lambda(x, y)(0). \]

Therefore, we deduce the following inequality:

\[ 4 |(1-\lambda)x + \lambda y| \cdot |\lambda x + (1-\lambda)y| \geq 4xy, \]

which is true, from Lemma 2.1.

We assume it is true for every integer \( \leq n \), so

\[ F_\lambda(x, y)(n+1) \geq F_\lambda(x, y)(n) \]

We will prove that

\[ F_\lambda(x, y)(n+2) \geq F_\lambda(x, y)(n+1). \quad (2.5) \]

Using the substitutions \( x \to (1-\lambda)x + \lambda y \) and \( y \to \lambda x + (1-\lambda)y \) in the inequality (2.3), we deduce

\[ F_\lambda((1-\lambda)x + \lambda y, \lambda x + (1-\lambda)y)(n) \geq \]

\[ \geq F_\lambda((1-\lambda)x + \lambda y, \lambda x + (1-\lambda)y)(n), \]

so, from equality (2.2), we have

\[ F_\lambda(x, y)(n+2) \geq F_\lambda(x, y)(n+1). \]

so we obtain (2.6).

According to inequality (2.3), we can write the sequence of inequalities

\[ F_\lambda(x, y)(n) \geq F_\lambda(x, y)(n-1) \geq \ldots \geq F_\lambda(x, y)(1) \geq F_\lambda(x, y)(0) = 4xy. \]

Therefore, we have

\[ F_\lambda(x, y)(n) \geq 4xy, \text{ for any } \lambda \in [0, 1], x, y \geq 0 \text{ and for all integers } n \geq 0. \]
If $\lambda \in (0, 1)$, then $1 - 2\lambda \in (-1, 1)$ and passing to limit when $n \to \infty$, we obtain

$$\lim_{n \to \infty} F_\lambda (x, y) (n) = (x + y)^2.$$ 

Since the sequence $(F_\lambda (x, y) (n))_{n \geq 0}$ is increasing, we deduce

$$(x + y)^2 \geq F_\lambda (x, y) (n),$$

for any $\lambda \in (0, 1), x, y \geq 0$ and for all integers $n \geq 0$.

From the inequalities above we have that

$$(x + y)^2 \geq F_\lambda (x, y) (n) \geq 4xy,$$

for any $\lambda \in (0, 1), x, y \geq 0$ and for all integers $n \geq 0$.

If $\lambda = 0$ and $\lambda = 1$, then $F_\lambda (x, y) (n) = 4xy$, so

$$(x + y)^2 \geq F_\lambda (x, y) (n) \geq 4xy.$$

It follows that

$$(x + y)^2 \geq F_\lambda (x, y) (n) \geq 4xy,$$

for any $\lambda \in [0, 1], x, y \geq 0$ and for all integers $n \geq 0$.

Thus, the proof of Theorem 2.2 is complete.

**Remark 1.** It is easy to see that there is the sequence of inequalities

$$(x + y)^2 \geq ... \geq F_\lambda (x, y) (n) \geq F_\lambda (x, y) (n - 1) \geq ...$$

$$... \geq F_\lambda (x, y) (1) \geq F_\lambda (x, y) (0) = 4xy.$$

(2.6)

**Corollary 2.3.** There are the following inequalities:

$$x + y \geq \sqrt{F_\lambda (x, y) (n)} \geq 2\sqrt{xy};$$

(2.7)

$$x^2 + y^2 \geq \sqrt{F_\lambda (x^2, y^2) (n)} \geq 2xy;$$

(2.8)

$$x + y + z \geq \frac{1}{2} \sum_{\text{cyclic}} \sqrt{F_\lambda (x, y) (n)} \geq \sqrt{xy} + \sqrt{yz} + \sqrt{zx};$$

(2.9)
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\[ x^2 + y^2 + z^2 \geq \frac{1}{2} \sum_{cyclic} F_\lambda (x, y) (n) \geq xy + yz + zx; \]  
(2.10)

\[ x^2 + y^2 + z^2 + xy + yz + zx \geq \frac{1}{2} \sum_{cyclic} F_\lambda (x, y) (n) \geq 2 (xy + yz + zx) \]  
(2.11)

and

\[ (x + y) (y + z) (z + x) \geq \prod_{cyclic} F_\lambda (x, y) (n) \geq 8xyz, \]  
(2.12)

for any \( \lambda \in [0, 1] \), for any \( x, y \geq 0 \), and for all integers \( n \geq 0 \).

**Proof.** From Theorem 2.2, we easily deduce inequality (2.7). Using the substitutions \( x \rightarrow x^2 \) and \( y \rightarrow y^2 \) in inequality (2.7), we obtain inequality (2.8). Similarly to inequality (2.7), \( x + y \geq \sqrt{F_\lambda (x, y) (n)} \geq 2 \sqrt{xy} \), we can write the following inequalities:

\[ y + z \geq \sqrt{F_\lambda (y, z) (n)} \geq 2 \sqrt{yz} \]  
and

\[ z + x \geq \sqrt{F_\lambda (z, x) (n)} \geq 2 \sqrt{zx} \],

which means, by adding, that

\[ x + y + z \geq \frac{1}{2} \sum_{cyclic} \sqrt{F_\lambda (x, y) (n)} \geq \sqrt{xy} + \sqrt{yz} + \sqrt{zx}. \]

It is easy to see that, by making the substitutions \( x \rightarrow x^2 \) and \( y \rightarrow y^2 \) in inequality (2.9), we obtain inequality (2.10). Similar to inequality (2.4), \( (x + y)^2 \geq F_\lambda (x, y) (n) \geq 4xy \), we obtain the following inequalities:

\[ (y + z)^2 \geq F_\lambda (y, z) (n) \geq 4yz \]  
and

\[ (z + x)^2 \geq F_\lambda (z, x) (n) \geq 4zx. \]

By adding them, we have inequality (2.11) and by multiplying them, we obtain inequality (2.12).

**Lemma 2.4** For any triangle ABC, the following inequality,

\[ \sqrt{ab} + \sqrt{bc} + \sqrt{ca} \geq \frac{4\Delta}{R}, \]  
(2.13)

holds.

**Proof.** We apply the arithmetic-geometric mean inequality and we find that

\[ \sqrt{ab} + \sqrt{bc} + \sqrt{ca} \geq 3 \sqrt[3]{abc}. \]
It is sufficient to show that
\[ 3\sqrt{abc} \geq \frac{4\Delta}{R}. \]  
(2.14)

Inequality (2.14) is equivalent to
\[ 27abc \geq \frac{64\Delta^3}{R^3}, \]
so
\[ 27 \cdot 4R\Delta \geq \frac{64\Delta^3}{R^3}, \]
which means that
\[ 27R^4 \geq 16\Delta^2. \]  
(2.15)

Using Mitrinović's Inequality, \(3\sqrt{3}R \geq 2s\), and Euler's Inequality \(R \geq 2r\), we deduce, by multiplication, that
\[ 3\sqrt{3}R^2 \geq 4\Delta. \]
It follows (2.15).

**Corollary 2.5.** In any triangle ABC, there are the following inequalities:

\[
R \geq \frac{4}{\sum_{cyclic} \sqrt{F_\lambda \left( \frac{1}{h_a}, \frac{1}{h_b} \right) (n)}} \geq 2r; \tag{2.16}
\]

\[
s \geq \frac{1}{2} \sum_{cyclic} \sqrt{F_\lambda (s-a, s-b) (n)} \geq 3\sqrt{3}r \tag{2.17}
\]

and
\[
a^{2\alpha} + b^{2\alpha} + c^{2\alpha} \geq \frac{1}{2} \sum_{cyclic} \sqrt{F_\lambda \left( a^{2\alpha}, b^{2\alpha} \right) (n)} \geq 3 \left( \frac{4\Delta}{\sqrt{3}} \right)^\alpha, \tag{2.18}
\]

for any \( \lambda \in [0,1], x, y \geq 0, n \geq 0 \) and \( \alpha \) is a real numbers.

**Proof.** Making the substitutions \( x = \frac{1}{h_a}, y = \frac{1}{h_b} \) and \( z = \frac{1}{h_c} \) in inequality (2.9), we obtain

\[
\frac{1}{h_a} + \frac{1}{h_b} + \frac{1}{h_c} \geq \frac{1}{2} \sum_{cyclic} \sqrt{F_\lambda \left( \frac{1}{h_a}, \frac{1}{h_b} \right) (n)} \geq \frac{1}{\sqrt{h_ah_b}} + \frac{1}{\sqrt{h_bh_c}} + \frac{1}{\sqrt{h_ch_a}}. \tag{2.19}
\]
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According to the equalities
\[ h_a = \frac{2\Delta}{a}, h_b = \frac{2\Delta}{b} \text{ and } h_c = \frac{2\Delta}{c}, \]
we have
\[ \frac{1}{\sqrt{h_a h_b}} + \frac{1}{\sqrt{h_b h_c}} + \frac{1}{\sqrt{h_c h_a}} = \frac{1}{2\Delta} \left( \sqrt{ab} + \sqrt{bc} + \sqrt{ca} \right). \]
From Lemma 2.4, we deduce
\[ \frac{1}{\sqrt{h_a h_b}} + \frac{1}{\sqrt{h_b h_c}} + \frac{1}{\sqrt{h_c h_a}} \geq \frac{2}{R}. \]
If we use the identity
\[ \frac{1}{h_a} + \frac{1}{h_b} + \frac{1}{h_c} = \frac{1}{r} \]
and inequality from above then inequality (2.18) becomes
\[ \frac{1}{r} \geq \frac{1}{2} \sum_{cyclic} \sqrt{F_\lambda \left( \frac{1}{h_a}, \frac{1}{h_b} \right)} (n) \geq \frac{2}{R}. \quad (2.20) \]
Consequently the inequalities (2.16) follows.

If in inequality (2.9) we take \( x = s - a, y = s - b \) and \( z = s - c \), then we deduce the inequality
\[ s \geq \frac{1}{2} \sum_{cyclic} \sqrt{F_\lambda (s-a, s-b)} (n) \geq \sqrt{(s-a)(s-b)} + \sqrt{(s-b)(s-c)} + \sqrt{(s-c)(s-a)}. \quad (2.21) \]
But, we know the identity \( \sum_{cyclic} \sqrt{(s-a, s-b)} = \sum_{cyclic} \sqrt{bc} \sin \frac{A}{2} \).

Using the arithmetic-geometric mean inequality, we obtain
\[ \sum_{cyclic} \sqrt{bc} \sin \frac{A}{2} \geq 3 \sqrt[3]{abc} \sin \frac{A}{2} \frac{B}{2} \sin \frac{C}{2} = 3 \sqrt[3]{4R\Delta r} = \frac{r}{4R} = 3\sqrt{\Delta r} \geq 3\sqrt[3]{3r^3} = 3\sqrt{3r}. \]
Hence,
\[ \sqrt{(s-a)(s-b)} + \sqrt{(s-b)(s-c)} + \sqrt{(s-c)(s-a)} \geq 3\sqrt{3}r, \quad (2.22) \]

which means, according to inequalities (2.21) and (2.22), that

\[ s \geq \frac{1}{2} \sum_{\text{cyclic}} \sqrt{F_{\lambda}(s-a, s-b)} \geq 3\sqrt{3}r. \]

Making the substitutions \( x = a^\alpha, y = b^\alpha, \) and \( z = c^\alpha \) in inequality (2.9), we obtain the following inequality:

\[ a^{2\alpha} + b^{2\alpha} + c^{2\alpha} \geq \frac{1}{2} \sum_{\text{cyclic}} \sqrt{F_{\lambda}(a^{2\alpha}, b^{2\alpha})} \geq a^\alpha b^\alpha + b^\alpha c^\alpha + c^\alpha a^\alpha. \quad (2.23) \]

Applying the arithmetic-geometric mean inequality and Pólya-Szegő's Inequality, \( \sqrt[3]{a^2 b^2 c^2} \geq \frac{4\Delta}{\sqrt[3]{3}} \), we deduce

\[ a^\alpha b^\alpha + b^\alpha c^\alpha + c^\alpha a^\alpha \geq \frac{3}{4} \sqrt{(a^2 b^2 c^2)^\alpha} = 3 \left( \frac{\sqrt[3]{a^2 b^2 c^2}}{\sqrt[3]{3}} \right)^\alpha \geq 3 \left( \frac{4\Delta}{\sqrt[3]{3}} \right)^\alpha, \]

so

\[ a^\alpha b^\alpha + b^\alpha c^\alpha + c^\alpha a^\alpha \geq 3 \left( \frac{4\Delta}{\sqrt[3]{3}} \right)^\alpha. \quad (2.24) \]

According to inequalities (2.23) and (2.24), we obtain the inequality

\[ a^{2\alpha} + b^{2\alpha} + c^{2\alpha} \geq \frac{1}{2} \sum_{\text{cyclic}} \sqrt{F_{\lambda}(a^{2\alpha}, b^{2\alpha})} \geq 3 \left( \frac{4\Delta}{\sqrt[3]{3}} \right)^\alpha. \]

Thus, the statement is true.

**Remark 2.**

a) Inequality (2.16) implies the sequence of inequalities

\[
R \geq \frac{4}{\sum_{\text{cyclic}} \sqrt{F_{\lambda}(\frac{1}{n_a}, \frac{1}{n_b})}} \geq ... \geq \frac{4}{\sum_{\text{cyclic}} \sqrt{F_{\lambda}(\frac{1}{n_a}, \frac{1}{n_b})}} \geq \frac{4}{\sum_{\text{cyclic}} \sqrt{F_{\lambda}(\frac{1}{n_a}, \frac{1}{n_b})}} \geq ... \geq 2r \quad (2.25)
\]

b) For \( \alpha = 1 \) in inequality (2.18), we obtain
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\[ a^2 + b^2 + c^2 \geq \frac{1}{2} \sum_{\text{cyclic}} \sqrt{F_{\lambda}(a^2, b^2)}(n) \geq 4\sqrt{3}\Delta, \quad (2.26) \]

which proves Weitzenböck’s Inequality, namely

\[ a^2 + b^2 + c^2 \geq 4\sqrt{3}\Delta. \]

**Corollary 2.6.** For any triangle ABC, there are the following inequalities:

\[ 2(s^2 - r^2 - 4Rr) \geq \frac{1}{2} \sum_{\text{cyclic}} \sqrt{F_{\lambda}(a^2, b^2)}(n) \geq s^2 + r^2 + 4Rr, \quad (2.27) \]

\[ s^2 - 2r^2 - 8Rr \geq \frac{1}{2} \sum_{\text{cyclic}} \sqrt{F_{\lambda}((s-a)^2, (s-b)^2)}(n) \geq r(4R + r), \quad (2.28) \]

\[ \frac{(s^2 + r^2 + 4Rr)^2 - 8s^2Rr}{4R^2} \geq \frac{1}{2} \sum_{\text{cyclic}} \sqrt{F_{\lambda}(h_a^2, h_b^2)}(n) \geq \frac{2s^2r}{R}, \quad (2.29) \]

\[ (4R + r)^2 - 2s^2 \geq \frac{1}{2} \sum_{\text{cyclic}} \sqrt{F_{\lambda}(r_a^2, r_b^2)}(n) \geq s^2, \quad (2.30) \]

\[ \frac{8R^2 + r^2 - s^2}{8R^2} \geq \frac{1}{2} \sum_{\text{cyclic}} \sqrt{F_{\lambda}\left(\frac{\sin^4 A}{2}, \frac{\sin^4 B}{2}\right)}(n) \geq \frac{s^2 + r^2 - 8Rr}{16R^2} \quad (2.31) \]

and

\[ \frac{(4R + r)^2 - s^2}{4R^2} \geq \frac{1}{2} \sum_{\text{cyclic}} \sqrt{F_{\lambda}\left(\frac{\cos^4 A}{2}, \frac{\cos^4 B}{2}\right)}(n) \geq \frac{s^2 + (4R + r)^2}{8R^2}. \quad (2.32) \]

**Proof.** According to Corollary 2.3 we have the inequality

\[ x^2 + y^2 + z^2 \geq \frac{1}{2} \sum_{\text{cyclic}} \sqrt{F_{\lambda}(x^2, y^2)}(n) \geq xy + yz + zx. \]

Using the substitutions
Corollary 2.7. In any triangle $ABC$, there are the following inequalities:

\[
3s^2 - r^2 - 4Rr \geq \frac{1}{2} \sum_{cyclic} F_{\lambda}(a,b)(n) \geq 2 \left( s^2 + r^2 + 4Rr \right), \tag{2.33}
\]

\[
s^2 - r^2 - 4Rr \geq \frac{1}{2} \sum_{cyclic} F_{\lambda}(s-a,s-b)(n) \geq 2r \left( 4R + r \right), \tag{2.34}
\]

\[
\frac{(s^2 + r^2 + 4Rr)^2 - 8s^2Rr}{4R^2} \geq \frac{1}{2} \sum_{cyclic} F_{\lambda}(h_a,h_b)(n) \geq \frac{4s^2r}{R}, \tag{2.35}
\]

and

\[
(4R + r)^2 - s^2 \geq \frac{1}{2} \sum_{cyclic} F_{\lambda}(r_a,r_b)(n) \geq 2s^2 \tag{2.36}
\]

Proof. According to Corollary 2.3, we have the inequality

\[
x^2 + y^2 + z^2 + xy + yz + zx \geq \frac{1}{2} \sum_{cyclic} F_{\lambda}(x,y)(n) \geq 2 \left( xy + yz + zx \right). \]

Using the substitutions

\[
(x, y, z) \in \{(a, b, c), (s - a, s - b, s - c), (h_a, h_b, h_c), (r_a, r_b, r_c)\}
\]

we deduce the inequalities from the statement.

Corollary 2.8. For any triangle $ABC$ there are the following inequalities:

\[
2s \left( s^2 + r^2 + 2Rr \right) \geq \prod_{cyclic} \sqrt{F_{\lambda}(a,b)(n)} \geq 32sRr, \tag{2.37}
\]

\[
4sRr \geq \prod_{cyclic} \sqrt{F_{\lambda}((s-a),(s-b))(n)} \geq 8sr^2, \tag{2.38}
\]
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\[ \frac{s^2 r (s^2 + r^2 + 4Rr)}{R^2} \geq \prod_{\text{cyclic}} \sqrt{F_\lambda (h_a, h_b) (n)} \geq \frac{16s^2 r^2}{R}, \tag{2.39} \]

\[ 4s^2 R \geq \prod_{\text{cyclic}} \sqrt{F_\lambda (r_a, r_b) (n)} \geq 8s^2 r, \tag{2.40} \]

\[ \frac{(2R - r)(s^2 + r^2 - 8Rr) - 2Rr^2}{32R^3} \geq \prod_{\text{cyclic}} \sqrt{F_\lambda \left( \frac{\sin^2 A}{2}, \frac{\sin^2 B}{2} \right) (n)} \geq \frac{r^2}{2R^2} \tag{2.41} \]

and

\[ \frac{(4R + r)^3 + s^2 (2R + r)}{32R^3} \geq \prod_{\text{cyclic}} \sqrt{F_\lambda \left( \frac{\cos^2 A}{2}, \frac{\cos^2 B}{2} \right) (n)} \geq \frac{s^2}{2R^2} \tag{2.42} \]

**Proof.** According to Corollary 2.3, we have the inequality

\[ (x + y) (y + z) (z + x) \geq \prod_{\text{cyclic}} F_\lambda (x, y) (n) \geq 8xyz. \]

Using the substitutions

\[ (x, y, z) \in \{ \{a, b, c\}, \{s - a, s - b, s - c\}, \{h_a, h_b, h_c\}, \{r_a, r_b, r_c\}, \]

\[ \left( \frac{\sin^2 A}{2}, \frac{\sin^2 B}{2}, \frac{\sin^2 C}{2} \right), \left( \frac{\cos^2 A}{2}, \frac{\cos^2 B}{2}, \frac{\cos^2 C}{2} \right) \}, \]

we deduce the inequalities required.

**Remark 3.** From Corollary 2.7, we obtain the inequality

\[ 2s (s^2 + r^2 + 2Rr) \geq \prod_{\text{cyclic}} \sqrt{F_\lambda (a, b) (n)} \geq 32sRr \geq \]

\[ \geq 8 \prod_{\text{cyclic}} \sqrt{F_\lambda ((s - a), (s - b)) (n)} \geq 64s a^2. \]
We consider the expression
\[ G(x, y) (n) = \frac{xy (x^{n-1} + y^{n-1}) (x^{n+1} + y^{n+1})}{(x^n + y^n)^2}, \] (2.43)
where \( x, y > 0 \) and for all integers \( n \geq 0 \).

**Theorem 2.9.** For any \( x, y > 0 \) and for all integers \( n \geq 0 \), there are the following relations:

\[ a) \left( \frac{x + y}{2} \right)^2 \geq G(x, y) (n) \geq xy \] (2.44)

and

\[ b) G(x, y) (n + 1) \leq G(x, y) (n). \] (2.45)

**Proof.** We take \( \lambda = \frac{x}{x^n + y^n} \), for all integers \( n \geq 0 \), in inequality (2.1), because \( \lambda \in (0, 1) \), and we deduce

\[ \left( \frac{x + y}{2} \right)^2 \geq \frac{xy (x^{n-1} + y^{n-1}) (x^{n+1} + y^{n+1})}{(x^n + y^n)^2} \geq xy, \]

so,

\[ \left( \frac{x + y}{2} \right)^2 \geq G(x, y) (n) \geq xy. \]

To prove inequality (2.45), we can write

\[
\frac{G(x, y) (n + 1)}{G(x, y) (n)} - 1 = -\frac{(xy)^{n-1} (x - y)^2 (x^2 + xy + y^2) (x^{2n} + x^{2n-1}y + x^{2n}y^2 + \ldots + y^{2n})}{(x^{n+1} + y^{n+1})^3 (x^{n-1} + y^{n-1})} \leq 0.
\]

Consequently, we have

\[ G(x, y) (n + 1) \leq G(x, y) (n). \]

**Remark 4.** It is easy to see that there is the sequence of inequalities

\[ \left( \frac{x + y}{4} \right)^2 = G(x, y) (0) \geq G(x, y) (1) \geq \ldots \]
\[ \geq G(x, y)(n - 1) \geq G(x, y)(n) \geq \ldots \geq xy. \] (2.46)

**Corollary 2.10**

There are the following inequalities:

\[ \frac{x + y}{2} \geq \sqrt{G(x, y)(n)} \geq \sqrt{xy}; \] (2.47)

\[ \frac{x^2 + y^2}{2} \geq \sqrt{G(x^2, y^2)(n)} \geq \sqrt{xy}; \] (2.48)

\[ \frac{x + y + z}{2} \geq \sqrt{G(x, y)(n)} \geq \sqrt{xy} \quad \text{and} \quad \sqrt{y} + \sqrt{z} \geq \sqrt{xyz}; \] (2.49)

\[ \frac{x^2 + y^2 + z^2}{2} \geq \sqrt{G(x^2, y^2)(n)} \geq xy + yz + zx; \] (2.50)

\[ \frac{1}{2} \left( x^2 + y^2 + z^2 + xy + yz + zx \right) \geq \sum_{cyclic} G(x, y)(n) \geq xy + yz + zx \] (2.51)

and

\[ \frac{1}{8} (x + y)(y + z)(z + x) \geq \left( \prod_{cyclic} G(x, y)(n) \right) \geq xyz, \] (2.52)

for any \( x, y > 0 \) and for all integers \( n \geq 0 \).

**Proof.** From Theorem 2.9, we easily deduce inequality (2.47). Using the substitutions \( x \rightarrow x^2 \) and \( y \rightarrow y^2 \) in inequality (2.47), we obtain inequality (2.48). Similarly to inequality (2.47), \( \frac{x^2 + y^2}{2} \geq \sqrt{G(x, y)(n)} \geq \sqrt{xy} \), we can write the following inequalities:

\[ \frac{y + z}{2} \geq \sqrt{G(y, z)(n)} \geq \sqrt{yz} \quad \text{and} \quad \frac{z + x}{2} \geq \sqrt{G(z, x)(n)} \geq \sqrt{zx}, \]

which means, by adding, that

\[ x + y + z \geq \sum_{cyclic} \sqrt{G(x, y)(n)} \geq \sqrt{xy} + \sqrt{yz} + \sqrt{zx}. \]

It is easy to see that by making the substitutions \( x \rightarrow x^2 \) and \( y \rightarrow y^2 \) in inequality (2.49), we obtain inequality (2.50). Similarly to inequality (2.44), \( \left( \frac{x + y}{2} \right)^2 \geq G(x, y)(n) \geq xy \), we obtain the following inequalities:
\[
\left( \frac{y + z}{2} \right)^2 \geq G(y, z)(n) \geq yz \quad \text{and} \quad \left( \frac{z + x}{2} \right)^2 \geq G(z, x)(n) \geq zx.
\]

By adding them, we have inequality (2.51) and by multiplying them, we obtain inequality (2.52).

**Corollary 2.11.** In any triangle \(ABC\), there are the following inequalities:

\[
R \geq \frac{2}{\sum_{cyclic} \sqrt{G\left(\frac{1}{h_a}, \frac{1}{h_b}\right)}(n)} \geq 2r; \quad (2.53)
\]

\[
s \geq \sum_{cyclic} \sqrt{G(s - a, s - b)}(n) \geq 3\sqrt{3}r \quad (2.54)
\]

and

\[
a^{2\alpha} + b^{2\alpha} + c^{2\alpha} \geq \sum_{cyclic} \sqrt{G(a^{2\alpha}, b^{2\alpha})}(n) \geq 3\left(\frac{4\Delta}{\sqrt{3}}\right)^{\alpha}, \quad (2.55)
\]

for any \(n \geq 0\) and for every real numbers \(\alpha\).

**Proof.** Making the substitutions \(x = \frac{1}{h_a}, y = \frac{1}{h_b}\) and \(z = \frac{1}{h_c}\) in inequality (2.49), we obtain

\[
\frac{1}{h_a} + \frac{1}{h_b} + \frac{1}{h_c} \geq \sum_{cyclic} \sqrt{G\left(\frac{1}{h_a}, \frac{1}{h_b}\right)}(n) \geq \frac{1}{\sqrt{h_a h_b}} + \frac{1}{\sqrt{h_b h_c}} + \frac{1}{\sqrt{h_c h_a}}. \quad (2.56)
\]

From inequality (2.13), we have

\[
\frac{1}{\sqrt{h_a h_b}} + \frac{1}{\sqrt{h_b h_c}} + \frac{1}{\sqrt{h_c h_a}} \geq \frac{2}{R},
\]

and from the identity

\[
\frac{1}{h_a} + \frac{1}{h_b} + \frac{1}{h_c} = \frac{1}{r}
\]

we deduce

\[
\frac{1}{r} \geq \sum_{cyclic} \sqrt{G\left(\frac{1}{h_a}, \frac{1}{h_b}\right)}(n) \geq \frac{2}{R}. \quad (2.57)
\]
Consequently

\[ R \geq \frac{2}{\sum_{\text{cyclic}} \sqrt{G\left(\frac{1}{h_a}, \frac{1}{h_b}\right)}} (n) \geq 2r. \]

If in inequality (2.49) we take \( x = s - a, y = s - b \) and \( z = s - c \), then we deduce the inequality

\[ s \geq \sum_{\text{cyclic}} \sqrt{G(s - a, s - b)} (n) \geq \sqrt{(s - a)(s - b)} + \sqrt{(s - b)(s - c)} + \sqrt{(s - c)(s - a)}. \]

But

\[ \sqrt{(s - a)(s - b)} + \sqrt{(s - b)(s - c)} + \sqrt{(s - c)(s - a)} \geq 3\sqrt{3}r, \]

which means that

\[ s \geq \sum_{\text{cyclic}} \sqrt{G(s - a, s - b)} (n) \geq 3\sqrt{3}r. \]

Making the substitutions \( a^\alpha, b^\alpha, c^\alpha \), and in inequality (2.49), we obtain the following inequality:

\[ a^{2\alpha} + b^{2\alpha} + c^{2\alpha} \geq \sum_{\text{cyclic}} \sqrt{G(a^{2\alpha}, b^{2\alpha})} (n) \geq a^\alpha b^\alpha + b^\alpha c^\alpha + c^\alpha a^\alpha. \quad (2.59) \]

Applying the arithmetic-geometric mean inequality and Pólya-Szegő’s Inequality, \( \sqrt[\alpha]{a^{2\alpha}b^{2\alpha}c^{2\alpha}} \geq \frac{\Delta}{\sqrt{3}}, \) we deduce

\[ a^\alpha b^\alpha + b^\alpha c^\alpha + c^\alpha a^\alpha \geq \left(\frac{\Delta}{\sqrt{3}}\right)^\alpha. \]

Therefore

\[ s \geq \sum_{\text{cyclic}} \sqrt{G(s - a, s - b)} (n) \geq 3\sqrt{3}r. \]

**Remark 5.** a) Inequality (2.53) implies the sequence of inequalities

\[ R \geq \ldots \geq \frac{2}{\sum_{\text{cyclic}} \sqrt{G\left(\frac{1}{h_a}, \frac{1}{h_b}\right)}} (n) \geq \frac{2}{\sum_{\text{cyclic}} \sqrt{G\left(\frac{1}{h_a}, \frac{1}{h_b}\right)}} (n - 1) \geq \ldots \]
\[
\sum_{cyclic} \sqrt{\left( \frac{1}{h_a}, \frac{1}{h_b} \right)}(0) \geq 2r \tag{2.60}
\]

b) For \( \alpha = 1 \) in inequality (2.55), we obtain
\[
a^2 + b^2 + c^2 \geq \sum_{cyclic} \sqrt{G(a^2, b^2)}(n) \geq 4\sqrt{3}\Delta, \tag{2.61}
\]

which proves Weitzenböck’s Inequality, namely
\[
a^2 + b^2 + c^2 \geq 4\sqrt{3}\Delta.
\]

**Corollary 2.12.** For any triangle \( \triangle ABC \), there are the following inequalities:
\[
2(s^2 - r^2 - 4Rr) \geq \sum_{cyclic} \sqrt{G(a^2, b^2)}(n) \geq s^2 + r^2 + 4Rr, \tag{2.62}
\]
\[
s^2 - 2r^2 - 8Rr \geq \sum_{cyclic} \sqrt{G((s-a)^2, (s-b)^2)}(n) \geq r(4R + r), \tag{2.63}
\]
\[
\frac{(s^2 + r^2 + 4Rr)^2 - 8s^2Rr}{4R^2} \geq \sum_{cyclic} \sqrt{G(h_a^2, h_b^2)}(n) \geq \frac{2s^2r}{R}, \tag{2.64}
\]
\[
(4R + r)^2 - 2s^2 \geq \sum_{cyclic} \sqrt{G(r_a^2, r_b^2)}(n) \geq s^2 \tag{2.65}
\]
\[
\frac{8R^2 + r^2 - s^2}{8R^2} \geq \sum_{cyclic} \sqrt{G\left(\sin^2 \frac{A}{2}, \sin^2 \frac{B}{2}\right)}(n) \geq \frac{s^2 + r^2 - 8Rr}{16R^2} \tag{2.66}
\]
and
\[
\frac{4(R + r)^2 - s^2}{4R^2} \geq \sum_{cyclic} \sqrt{G\left(\cos^2 \frac{A}{2}, \cos^2 \frac{B}{2}\right)}(n) \geq \frac{s^2 + (4R + r)^2}{8R^2}. \tag{2.67}
\]

**Proof.** According to Corollary 2.10, we have the inequality
\[
x^2 + y^2 + z^2 \geq \sum_{cyclic} \sqrt{G(x^2, y^2)}(n) \geq xy + yz + zx.
\]
Using the substitutions

\[(x, y, z) \in \{(a, b, c), (s - a, s - b, s - c), (h_a, h_b, h_c), (r_a, r_b, r_c),\]
\[(\sin^2 \frac{A}{2}, \sin^2 \frac{B}{2}, \sin^2 \frac{C}{2}), (\cos^2 \frac{A}{2}, \cos^2 \frac{B}{2}, \cos^2 \frac{C}{2})\},\]

we deduce the inequalities required.

**Corollary 2.13.** In any triangle ABC there are the following inequalities:

\[
\frac{1}{2} (3s^2 - r^2 - 4Rr) \geq \sum_{cyclic} G(a, b) (n) \geq s^2 + r^2 + 4Rr, \tag{2.68}
\]

\[
\frac{1}{2} (s^2 - r^2 - 4Rr) \geq \sum_{cyclic} G(s - a, s - b) (n) \geq r (4R + r), \tag{2.69}
\]

\[
\left(\frac{s^2 + r^2 + 4Rr}{8R^2}\right)^2 \geq \sum_{cyclic} G(h_a, h_b) (n) \geq \frac{2s^2r}{R}, \tag{2.70}
\]

and

\[
\frac{1}{2} \left[(4R + r)^2 - s^2\right] \geq \sum_{cyclic} G(r_a, r_b) (n) \geq s^2. \tag{2.71}
\]

**Proof.** According to Corollary 2.10, we have the inequality

\[
\frac{1}{2} (x^2 + y^2 + z^2 + xy + yz + zx) \geq \sum_{cyclic} G(x, y) (n) \geq xy + yz + zx.
\]

Using the substitutions

\[(x, y, z) \in \{(a, b, c), (s - a, s - b, s - c), (h_a, h_b, h_c), (r_a, r_b, r_c)\},\]

we deduce the inequalities from the statement.

**Corollary 2.14.** For any triangle ABC there are the following inequalities:

\[
\frac{1}{4} s (s^2 + r^2 + 2Rr) \geq \prod_{cyclic} \sqrt{G(a, b)(n)} \geq 4sRr, \tag{2.72}
\]

\[
\frac{1}{2} sRr \geq \prod_{cyclic} \sqrt{G((s - a), (s - b))(n)} \geq sr^2, \tag{2.73}
\]
\[
\frac{s^2r (s^2 + r^2 + 4Rr)}{8R^2} \geq \prod_{\text{cyclic}} \sqrt{G(h_a, h_b)}(n) \geq \frac{2s^2r^2}{R}, \quad (2.74)
\]
\[
\frac{1}{2} s^2R \geq \prod_{\text{cyclic}} \sqrt{G(r_a, r_b)}(n) \geq s^2r, \quad (2.75)
\]
\[
\frac{(2R - r) (s^2 + r^2 - 8Rr) - 2Rr^2}{256R^3} \geq \prod_{\text{cyclic}} \sqrt{G\left(\frac{\sin^2 A}{2}, \frac{\sin^2 B}{2}\right)}(n) \geq \frac{r^2}{16R^2} \quad (2.76)
\]

and
\[
\frac{(4R + r)^3 + s^2 (2R + r)}{256R^3} \geq \prod_{\text{cyclic}} \sqrt{G\left(\frac{\cos^2 A}{2}, \frac{\cos^2 B}{2}\right)}(n) \geq \frac{s^2}{16R^2}. \quad (2.77)
\]

**Proof.** According to Corollary 2.10, we have the inequality
\[
\frac{1}{8} (x + y)(y + z)(z + x) \geq \prod_{\text{cyclic}} G(x, y)(n) \geq xyz.
\]
Using the substitutions
\[
(x, y, z) \in \{(a, b, c), (s - a, s - b, s - c), (h_a, h_b, h_c), (r_a, r_b, r_c), (\frac{\sin^2 A}{2}, \frac{\sin^2 B}{2}, \frac{\sin^2 C}{2}), (\frac{\cos^2 A}{2}, \frac{\cos^2 B}{2}, \frac{\cos^2 C}{2})\},
\]
we deduce the inequalities required.

**Remark 6.** From Corollary 2.13, we obtain the inequality
\[
\frac{1}{4} s (s^2 + r^2 + 2Rr) \geq \prod_{\text{cyclic}} \sqrt{G(a, b)}(n) \geq 4sRr \geq \prod_{\text{cyclic}} \sqrt{G((s - a), (s - b))}(n) \geq 8sr^2. \quad (2.78)
\]
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