An extension of Ky Fan’s inequality

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ABSTRACT. We offer a generalization of the inequality $\frac{A_n}{A'_n} \geq \frac{G_n}{G'_n}$, due to Ky Fan.

Let $A_n, G_n$ denote the arithmetic, respectively geometric means and of $x_i$ ($i = 1, n$), where $x_i \in (0, \frac{1}{2}]$. Put $A'_n = A'_n (x_i) = A_n (1 - x_i)$; $G'_n = G'_n (x_i) = G_n (1 - x_i)$. The famous inequality of Ky Fan (see e.g. [1]) states that

$$\frac{A_n}{A'_n} \geq \frac{G_n}{G'_n} \quad (1)$$

Define $A_k = A_k (a_i) = \left(\frac{a_1 + \ldots + a_n}{n}\right)^{1/k}$ for $k \neq 0$ and $a_i > 0$ ($i = 1, n$), while $A_0 = A_o (a_i) = \lim_{k \to 0} A_k = \sqrt[n]{a_1 \ldots a_n} = G_n (a) = G_n$.

Then the following extension of (1) holds true:

**Theorem.** For all $x_i \in (0, \frac{1}{2}]$ and all $k \in R$ one has

$$1 + \left[\left(\frac{1-x_1}{x_1}\right)^{k} + \ldots + \left(\frac{1-x_n}{x_n}\right)^{k}\right] \cdot \frac{1}{n} \geq \left(\frac{1}{x_1^{k-1}} + \ldots + \frac{1}{x_n^{k-1}}\right)^{\frac{1}{k-1}} \quad (2)$$

**Proof.** Remark that for $k = O$, inequality (2) becomes

$$\sqrt[n]{1 - x_1} \ldots 1 - x_n \geq \frac{x_1 + \ldots + x_n}{n} - 1,$$

which is relation (1), i.e. the Ky Fan inequality.

Put now $\frac{1-x_i}{x_i} = a_i$ in relation (2). Since $x_i \in (0, \frac{1}{2}]$, we get $a_i \geq 1$.

After some elementary transformations, relation (2) becomes

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\[ 1 + \left( \frac{a_1^k + \ldots + a_n^k}{n} \right)^{\frac{1}{k}} \geq \left[ \frac{(a_1 + 1)^{2k-1} + \ldots + (a_n + 1)^{2k-1}}{n} \right]^{\frac{1}{2k-1}} \]  

(3)

where \( k \neq 0 \).

Let \( a_k = y_k^k (y_k > 0) \) and consider the application \( f (y) = \left( 1 + y^k \right)^{2k-1} \);

where \( y \geq 1 \). It is immediate that this function is convex for \( k < 0 \) or \( 0 < k \leq \frac{1}{2} \), or \( k \geq 1 \); and concave for \( k \in \left[ \frac{1}{2}, 1 \right] \).

Using the Jensen inequality for convex (concave) functions, relation (3) follows at one for \( k \neq \frac{1}{2} \).

For \( k = \frac{1}{2} \), however we have to prove the inequality

\[ 1 + \left( \frac{\sqrt{a_1} + \ldots + \sqrt{a_n}}{n} \right)^2 \geq \sqrt{(1 + a_1) \ldots (1 + a_n)} \]  

(4)

Letting \( a_k = y_k^2 \) and \( f (y) = \log (1 + y^2) \) which is concave, by Jensen’s inequality follows at one again (4).

REFERENCE


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