On Constantive Simple and Order-Primal Algebras

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Abstract. A finite algebra $A = (A; F_A)$ is said to be order-primal if its clone of all term operations is the set of all operations defined on $A$ which preserve a given partial order $\leq$ on $A$. In this paper we study algebraic properties of order-primal algebras for connected ordered sets $(A; \leq)$. Such order-primal algebras are constantive, simple and have no non-identical automorphisms. We show that in this case $F_A$ cannot have only unary fundamental operations or only one at least binary fundamental operation. We prove several properties of the varieties and the quasi-varieties generated by constantive and simple algebras and apply these properties to order-primal algebras. Further, we use the properties of order-primal algebras to formulate new primality criteria for finite algebras.

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1. Introduction

A clone $C$ on a set $A$ is a set of operations defined on $A$ which is closed under composition and contains all projections. If $A = (A; F_A)$ is an algebra, then the clone of all term operations $T(A)$ of $A$ is the clone which is generated by $F_A$. Let $O(A)$ be the clone of all operations defined on $A$. Finite algebras $A = (A; F_A)$ are called primal if $T(A) = O(A)$; that is, if every operation defined on $A$ is a term operation of $A$. Let $(A; \leq)$ be a finite partially ordered set and let $Pol(\leq)$ be the set of all operations defined on $A$ which preserve the relation $\leq$. Then a finite algebra $A = (A; F_A)$ is called order-primal if $T(A) = Pol(\leq)$. Order-primal algebras have received quite a bit of attention recently, see [3, 9, 11, 13]. A partially ordered set $(A; \leq)$ is called connected if for any two elements $a, b \in A$ there exist

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a natural number \( n \) and elements \( a = a_0, a_1, \ldots, a_n = b \) such that \( a_0 \leq a_1 \geq a_2 \leq \ldots \geq a_n (\leq a_n) = b \) (or \( a_0 \geq a_1 \leq a_2 \geq \ldots \geq a_n (\geq a_n) = b \)). Clearly, if \( (A; \leq) \) has the least element or the greatest element then \( (A; \leq) \) is connected. In [3] it was shown that for a connected finite ordered set \( (A; \leq) \) the order-primal algebra \( A = (A; F^d) \) has no proper subalgebra and is simple. Therefore, the only nontrivial algebra in \( HS\{A\} \) is isomorphic to \( A \). The variety \( V(\{A\}) \) generated by \( A \) is congruence distributive if and only if \( A \) is the only sub-directly irreducible algebra in \( V(\{A\}) \). In this case \( V(\{A\}) \) has no non-trivial sub-variety. If \( (A; \leq) \) is a finite bounded ordered set (that is, there exist the least and the greatest element with respect to \( \leq \)) then \( Pol(\leq) \) is one of the maximal sub-clones of the clone \( O(A) \) of all operations defined on \( A \). In this case the order-primal algebra \( A = (A; F^d) \) generates a minimal variety and has no nontrivial automorphism [5, 8].

If the bounded order \( \leq \) is given by the following Hasse diagram

![Hasse diagram]

then the variety generated by \( A \) has infinitely many sub-directly irreducible algebras [8] and therefore \( V(\{A\}) \) is not congruence distributive. But in this case the algebra \( A = (A; F^d) \) has no finite set \( F^d \) of fundamental operations [16]. In [4] it was shown that for fences the variety \( V(\{A\}) \) is congruence distributive and the algebra \( A \) can be represented by finitely many fundamental operations. If \( A \) is finite, there are only finitely many maximal subclones of \( O(\{A\}) \). Every proper subclone of \( O(\{A\}) \) is contained in a maximal one. Rosenberg [14] described all these maximal subclones by certain classes of relations. The \( n \)-ary operation \( f \in O(\{A\}) \) preserves the \( h \)-ary relation \( \rho \subseteq A^h \) if \( (a_{11}, \ldots, a_{1h}) \in \rho, \ldots, (a_{n1}, \ldots, a_{nh}) \in \rho \) implies \( (f(a_{11}, \ldots, a_{n1}), \ldots, f(a_{1h}, \ldots, a_{nh})) \in \rho \). By \( Pol(\rho) \) we denote the set of all operations which preserve \( \rho \). \( Pol(\rho) \) is always a clone. Every maximal subclone of \( O(\{A\}) \) can be described in this way using the following classes of relations:

**Class(1):** is the class of all bounded partial order relations on \( A \).

**Class(2):** is the class of all binary relations \( \{(a, s(a)) | a \in A\} \) where \( s \) is a permutation on \( A \) without invariant elements with all cycles of the same prime length.

**Class(3):** is the class of all quaternary relations \( \alpha \) defined by \( (a, b, c, d) \in \alpha \) if and only if \( a + b = c + d \) where \( + \) is the addition of an elementary abelian \( p \)-group (\( p \) is prime) on \( A \).
Class(4): is the class of all non-trivial equivalence relations on \( A \).

Class(5): is the class of all central, i.e., totally reflexive and totally symmetric \( h \)-ary relations \( r \) with \( 1 \leq h \leq |A| \) having a non-trivial center. A relation is called totally reflexive if it contains all \( h \)-tuples with a repetition of elements. An \( h \)-ary relation \( r \) is said to be totally symmetric if \((x_0, \ldots, x_{h-1}) \in r\) implies \((x_{s(0)}, \ldots, x_{s(h-1)}) \in r\) for any permutation \( s \) of \( \{0, \ldots, h-1\} \). The center of a totally symmetric and reflexive \( h \)-ary relation \( r \) is the set of all elements \( c \in A \) such that \((c, x_1, \ldots, x_{h-1}) \in r\) for all \( x_1, \ldots, x_{h-1} \in A \). For \( h = 1 \) the relation \( r \) is simply a subset of \( A \).

Class(6): is the class of all \( h \)-regularly generated relations which are defined for \( 3 \leq h \leq |A| \) by the following steps: for \( m \geq 1 \), \( m \in \mathbb{N} \) a set \( \theta_0, \ldots, \theta_{m-1} \) of equivalence relations on \( A \) is called \( h \)-regular if each \( \theta_i \), \( 0 \leq i \leq m-1 \) defines exactly \( h \) equivalence classes and if the intersection \( \bigcap_{i=1}^{m} \epsilon_i \) of arbitrary \( m \) equivalence classes \( \epsilon_i \) of \( \theta_i \) is nonempty. An \( h \)-ary relation \( \rho \) is said to be \( h \)-regularly generated associated with \( \theta_0, \ldots, \theta_{m-1} \) if \((a_1, \ldots, a_h) \in \rho\) if and only if for each \( 0 \leq i \leq m-1 \) at least two of the elements \( a_1, \ldots, a_h \) are equivalent modulo \( \theta_i \) (Recall that for \( h = |A| \) the clone generated by an \( h \)-regularly generated relation on \( A \) consists exactly of all unary and all non-surjective operations defined on \( A \). This clone is known under the name Słupecki clone).

Then in [13] was proved.

THEOREM 1.1 ([13]). For an unbounded connected ordered set \((A; \leq)\) the clone \( \text{Pol}(\leq) \) is not contained in a maximal clone \( \text{Pol}(\rho) \) where \( \rho \) is a relation from one of the classes (1) to (4).

Using the maximal clones, one gets the following

1.2. Primality Criterion: A finite algebra \( A = (A; F^A) \) is primal if and only if the clone \( T(A) \) is not contained in one of the maximal clones \( \text{Pol}(\rho) \) where \( \rho \) is a relation from one of the classes (1) to (6).

2. Properties of Order-Primal Algebras for Connected Orders

Let \( A = (A; F^A) \) be a finite algebra with \( T(A) = \text{Pol}(\leq) \) for a connected order \( \leq \) on \( A \). Then we have:

PROPOSITION 2.1. \( A \) is simple, has no proper subalgebra and no non-identical automorphism.

Proof. For bounded orders this is clear. We assume that \( \leq \) is unbounded. Then we can apply Theorem 1.1 to see that \( A \) is simple and has no non-identical automorphism. That \( A \) has no proper subalgebra is trivial, since for every \( a \notin A \) the constant operation \( c_a \) is a term operation of the order primal algebra \( A \).
We defined order-primal algebras as non-indexed algebras $A = (A; F_A)$. If we change to the indexed form $A = (A; (f^A_i)_{i \in I})$ where $f^A_i$ is $n_i$-ary, there arises the question for the type $\tau = (n_i)_{i \in I}$ of $A$. We recall the following primality criterion of Rousseau for algebras with a single at least binary fundamental operation.

**Theorem 2.2 ([15]).** A finite algebra $A = (A; f^A)$ ($|A| > 1$) with a single fundamental operation $f^A$, where $f^A$ is at least binary, is primal if and only if $A$ is simple, has no proper subalgebra and no non-identical automorphism.

Then Proposition 2.1 gives the following proposition:

**Proposition 2.3.** If $A = (A; F_A)$ is an order-primal algebra for a connected ordered set $(A; \leq)$, then $F_A$ contains at least two operations one of them at least binary.

**Proof.** If $F_A$ contains only unary operations, then the clone $T(A)$ of all term operations of $A$ consists also only of essentially unary operations. But for every order relation $\leq$ on $A$, there exist monotone operations which are at least essentially binary. Thus $T(A) \neq Pol(\leq)$, a contradiction. Assume now that $F_A = \{f^A\}$ for an at least binary operation. Since $A$ has no proper subalgebra, no non-identical automorphism and it is simple, by Theorem 2.2 it has to be primal; that is, $T(A) = Pol(\leq) = O(A)$. This is also a contradiction since $O(A)$ contains operations which are not monotone with respect to $\leq$. \(\square\)

The following concept is useful to describe the variety $V(A)$ which is generated by an order-primal algebra $A$.

**Definition 2.4 ([6]).** A non-trivial finite algebra $A = (A; F_A)$ is said to be semiframal if there exist two elements $a, b \in A$, $a \neq b$ and a binary term operation $f^A$ such that $f^A(a, y) = a$ and $f^A(b, y) = y$ for all $y \in A$.

Then we can prove:

**Proposition 2.5.** Let $A = (A; F_A)$, $|A| \geq 2$ be an order-primal algebra with respect to an order relation $\leq$ on $A$ with the least element $0 \in A$ or with the greatest element $1 \in A$. Then $A$ is semiframal.

**Proof.** Assume that there exists the least element $0$ with respect to $\leq$. Consider the binary operation $f^A$ defined by

$$f^A(x, y) = \begin{cases} 0, & x = 0 \\ y, & \text{otherwise}. \end{cases}$$
Then \( f^A(0, y) = 0 \) for all \( y \in A \). Since \(|A| \geq 2\), there is an element \( b \in A \) which is different from 0 and satisfies \( f^A(b, y) = y \) for all \( y \in A \). We have to show that \( f^A \) is a term operation of \( A \). Indeed, let \( x_1 \leq x_2 \) and \( y_1 \leq y_2 \). If \( x_2 = 0 \), then \( x_1 = 0 \) and \( f^A(x_1, y_1) = f^A(0, y_1) = 0 \leq 0 = f^A(0, y_2) = f^A(x_2, y_2) \). If \( x_2 \neq 0 \), then \( f^A(x_1, y_1) = 0 \leq y_2 = f^A(x_2, y_2) \) for the case that \( x_1 = 0 \) and \( f^A(x_1, y_1) = y_1 \leq y_2 = f^A(x_2, y_2) \) for the case that \( x_1 \neq 0 \). Therefore, \( f^A \) is monotone and hence \( f^A \) is a term operation of \( A \). For the greatest element, one can conclude in a similar way.

3. Varieties and Quasi-Varieties Generated by Constantive Simple and by Order-Primal Algebras for Connected Orders

If the variety \( V(A) \) which is generated by an order-primal algebra for a connected order is congruence distributive, then using B. Jonsson’s well-known Lemma, every \( B \in V(A) \) is isomorphic to a subdirect product of homomorphic images of subalgebras of \( A \). Since by Proposition 2.1 \( A \) is simple and has no proper subalgebra, \( A \) is the only sub-directly irreducible algebra and \( V(A) \) has no nontrivial subvariety (\( A \) is equationally maximal).

We can generalize this result using the following concept:

A finite non-trivial algebra \( A = (A; F^A) \) is called constantive if all constant operations are term operations of \( A \). Every order-primal algebra is constantive. Then by a result of Foster ([7]), we have.

LEMMA 3.1 ([7]). Let \( A \) be a finite nontrivial constantive and simple algebra. If \( A \) is semiframal, then \( V(A) \) is minimal.

Then Proposition 2.5 gives:

PROPOSITION 3.2. Let \( A = (A; F^A) \) be an order-primal algebra with respect to an order relation \( \leq \) on \( A \) with the least element \( 0 \in A \) or the greatest element \( 1 \in A \). Then \( V(A) \) is minimal.

For an arbitrary class \( K \) of algebras let \( Q(K) \) denote the smallest quasivariety containing \( K \). It is well known (see e.g., [2]) that \( Q(K) = \text{ISPP}_u(K) \), where the right-hand side denotes the class of all isomorphic copies of all subalgebras of \( K \) of products of ultraproducts. Clearly, we have \( V(K) = H(Q(K)) \), i.e., the variety generated by \( K \) is the class of all homomorphic images of algebras in \( Q(K) \). If \( A \) is an algebra, then \( Q(A) \) stands for the quasivariety \( Q(\{A\}) \).

If \( A_J^J \) is a direct power of \( A \), then to any \( a \in A \) there corresponds an element \( a_J^J \in A_J^J \) with \( \pi_J(a_J^J) = a \) for all \( j \in J \), where \( \pi_J \) are the projection mappings. The set \( \{a_J^J \mid a \in a \} \subseteq A_J^J \) is the universe of the so-called diagonal subalgebra \( \Delta \subseteq A_J^J \) which is isomorphic to \( A \).
PROPOSITION 3.3. Let \( A \) be a constantive and simple algebra. Then

(i) \( Q(A) \) is a minimal quasivariety.

(ii) If an algebra \( B \in V(A) \) fails to contain a subalgebra isomorphic to \( A \), then it contains a one-element subalgebra.

Proof.

(i) Take a nontrivial algebra \( B \in Q(A) \). Then \( B \in ISPP_\mu(\{A\}) \) and \( Q(B) \subseteq Q(A) \). Since \( A \) is finite, we have \( P_\mu(\{A\}) = \{A\} \), whence we get that \( B \in ISP(\{A\}) \). Hence, there exists a subalgebra \( C \) of a direct power \( A^l \) with \( B \cong C \). Since \( A \) is constantive, \( C \) contains the diagonal subalgebra \( \Delta \) of \( A^l \) as a subalgebra. As \( \Delta \cong A \), we get \( A \in IS(\{B\}) \subseteq Q(B) \) and this gives \( Q(A) = Q(B) \), proving the minimality of \( Q(A) \).

(ii) Let \( C \) be a nontrivial algebra of \( V(A) \). Then there exists a nontrivial algebra \( B \in Q(A) \) and a homomorphism \( h : B \rightarrow C \) such that \( h(B) = C \). In view of the proof of (i), \( B \) must contain a subalgebra \( D \) isomorphic to \( A \). As \( A \) is simple, the subalgebra \( h(D) \) of \( C \) is either a one-element algebra or we have \( h(D) \cong D \cong A \). So if the algebra \( C \) fails to contain a subalgebra isomorphic to \( A \), then it has a one element subalgebra, namely \( h(D) \).

A variety consisting of one-element algebras is called trivial or degenerate. If no algebra of a variety \( V \) consisting of at least two elements has a one-element subalgebra, the variety \( V \) is said to be semidegenerate ([12]). Now using Proposition 3.3 (ii) we can deduce the following result of [5]:

LEMMA 3.4 ([5]). If \( A \) is constantive and simple and \( V(A) \) is semidegenerate, then \( V(A) \) is minimal.

Proof. Let \( B \) be a nontrivial algebra of \( V(A) \). Since \( B \) fails to contain a one-element subalgebra, in view of Proposition 3.3(ii), it contains a subalgebra \( D \cong A \). Hence \( A \in V(B) \), and this implies that \( V(A) \) is minimal.

PROPOSITION 3.5. Let \( A \) be a constantive and simple algebra. Then the following are equivalent:

(i) \( V(A) \) is a minimal variety.

(ii) \( Q(A) \) is the least nontrivial quasivariety of \( V(A) \).

(iii) The algebra \( A \) can be embedded as a subalgebra in any non-trivial algebra \( B \in V(A) \).

Proof. (i) \( \Rightarrow \) (ii) Assume that \( V(A) \) is a minimal variety and take any subquasivariety \( Q \subseteq V(A) \). Then \( H(Q) = V(Q) = V(A) \). Hence, there is an algebra \( B \in Q \) such that \( A \) is a homomorphic image of \( B \). As \( A \) does not contain one-element subalgebras, the same must be valid for \( B \). Therefore, by applying Pro-
position 3.3(ii), we get that there exists a subalgebra $B'$ of $B$ with $B' \cong A$. Hence $Q(A) = Q(B') \subseteq Q$, proving (ii).

(ii) $\Rightarrow$ (iii) Let $B$ be a nontrivial algebra of $V(A)$. As $V(A)$ is locally finite, $B$ contains a finite nontrivial algebra $D$. Then $Q(D) = ISP(\{D\})$. Since by assumption $Q(A) \subseteq Q(D) = ISP(\{D\})$, there exists an algebra $A' \cong A$ with $A' \subseteq B'$ for some $B' \in ISP(\{D\})$. Then obviously, there is a subalgebra $A''$ of a direct power $D^J$ with $A'' \cong A$. Since $A''$ is nontrivial, there exists at least one projection $\pi_{i_0}$ of $A''$ into $D$ (where $i_0 \in J$) such that its image is not a one-element algebra. Since $A''$ is simple, we get $A \cong A'' \cong \pi_{i_0}(A'') \subseteq D \subseteq B$, proving (iii).

The implication (iii) $\Rightarrow$ (i) is straightforward. 

\[ \square \]

Since non-trivial order-primal algebras for connected orders are constantive and simple, Propositions 3.3 and 3.5 are satisfied for those algebras.

An important question is whether the variety generated by an order primal algebra $A$ can be congruence modular (congruence distributive) or congruence permutable. It is quite clear that it cannot be congruence permutable. Otherwise in $A$ there must exist a term $p^A$ such the identities $p(x, x, y) \approx y$ and $p(x, y, y) \approx x$ are satisfied in $A$. Assume that $a, b \in A$ with $a \leq b$ and $a \neq b$. Then from $a \leq a, a \leq b, b \leq b$ there follows $b = p(a, a, b) \leq p(a, b, b) = a$, a contradiction. It was proved in [5], that a finite algebra $A = (A, F^A)$ with $T(A) = \langle F^A >= Pol(\rho)$, where $\rho$ is a relation from Class(6), cannot generate a congruence distributive variety. But then a finite algebra $A'$ with $T(A') \subseteq T(A) = Pol(\rho)$ can also not generate a congruence distributive variety. Indeed, if $T(A)$ does not contain Jonsson’s terms, characterising congruence distributive varieties, then $T(A') \subseteq T(A)$ can also not contain these operations.

Let us now consider the congruence modular case. An important result of [18] is the fact that the variety $V(A)$ generated by an order-primal algebra is congruence modular if and only if it is congruence distributive. In [10] and [17] several equivalent conditions for the congruence modularity of $V(A)$ in the case of a finite connected poset $(A, \leq)$ are given. For instance, in [10] is proved that this is equivalent to the existence of a near unanimity function on $A$. In [17] several properties of the order variety generated by $(A; \leq)$, i.e. the smallest class of ordered sets which is closed under rejections, products and subposets which are preserved by every idempotent operation defined on $A$ and contains $(A; \leq)$, were studied. A finite poset $(A, \leq)$ is called dismantlable if its elements can be listed in such an order $A = \{x_1, \ldots, x_n\}$ that $x_i$ is irreducible in $\{x_i, \ldots, x_n\}$. For an equivalent definition see e.g. [19]. Then from a result in [17] one obtains that if $V(A)$ is congruence modular then the poset of $(A; \leq)$ is dismantlable.

Using the characterization of minimal, locally finite varieties given in [20], B. Larose ([11]) characterized recently all order primal algebras for connected orders which generate a minimal variety.
THEOREM 3.6 ([11]). Let $A$ be an order-primal algebra for a connected order and $|A| \geq 2$. Then $V(A)$ is minimal if and only if $(A; \leq)$ is dismantlable.

If for an order-primal algebra $A$ for a connected order the variety $V(A)$ is congruence modular, then by [17] the order $(A; \leq)$ is dismantlable and therefore $V(A)$ is minimal. But we have also:

PROPOSITION 3.7. Let $A$ be an order-primal algebra for a connected ordered set $(A, \leq)$ with $|A| \geq 2$. If $V(A)$ is a congruence modular variety, then it is a minimal semidegenerate quasivariety.

Proof. As we have pointed out before, $V(A)$ is minimal and its only subdirectly irreducible (nontrivial) algebra is $A$. Since in this case any nontrivial algebra of $V(A)$ is isomorphic to a subdirect power of $A$, we conclude that $V(A) = Q(A)$. Hence $V(A)$ is a minimal quasivariety, according to Proposition 3.3(1). Now, suppose that there is a non-trivial algebra $B$ of $V(A)$ with a one-element subalgebra $\{e\}$. Since $B$ is a subdirect power of $A$, there is a homomorphism $h : B \to A$. Hence we get that $\{h(e)\}$ is a one-element subalgebra of $A$, a contradiction. $\square$

A quasivariety $Q$ is called structurally complete if for every subquasivariety $Q'$, which is properly contained in $Q$ the variety $V(Q')$ is properly contained in the variety $V(Q)$ (see [1]). For a variety $V$ the above notion can be formulated similarly: $V$ is structurally complete if and only if any proper subquasivariety of it generates a proper subvariety. Obviously, any minimal quasivariety is structurally complete. It is interesting, that in our case even the converse of this statement is true. In fact, we have the following:

COROLLARY 3.8. If $A$ is an order-primal algebra for a connected ordered set $(A, \leq)$ of finite type and with $|A| \geq 2$, then $V(A)$ is structurally complete if and only if it is a minimal quasivariety.

Proof. Assume that $V(A)$ is structurally complete. Then from $Q(A) \subset V(A)$ there follows $V(A) = V(Q(A)) \subset V(A)$ – a contradiction. Thus $V(A) = Q(A)$ and hence Lemma 3.3(i) gives that $V(A)$ is a minimal quasivariety. The converse implication is well-known. $\square$

4. Primality Criteria

If $A$ is an order-primal algebra for a connected unbounded order, then by Theorem 1.1 the clone $T(A)$ of all term operations of $A$ is not contained in a maximal clone $Pol(\rho)$ where $\rho$ is a relation from one of the classes (1) to (4).
Since \( A \) is not primal, \( T(A) \) is contained in \( Pol(\rho) \) where \( \rho \) belongs to Class(5) or to Class(6). We consider the following cases:

1. \( T(A) \subseteq Pol(\rho) \), \( \rho \) belongs to Class(6). Then by the remark in Section 3, the algebra \( A \) cannot generate a congruence distributive (congruence modular) variety.

2. \( T(A) \not\subseteq Pol(\rho) \), \( \rho \) belongs to Class (6). Then \( T(A) \subseteq Pol(\rho) \) for a relation \( \rho \) belonging to Class(5).

Now we assume that \( A = (A, F^A) \) is an algebra with \( T(A) \not\supseteq Pol(\rho) \) for a connected unbounded order \( \leq \) on \( A \).

Then using Theorem 1.1 we obtain the following primality criteria:

**COROLLARY 4.1.** If \( A = (A; F^A) \) is a finite algebra with \( T(A) \supseteq Pol(\leq) \) for a connected unbounded order on \( A \), then \( A \) is primal if and only if \( A \) generates a congruence permutable variety \( V(A) \).

**Proof.** If \( A \) is primal then \( V(A) \) is congruence permutable. If conversely \( V(A) \) is congruence permutable then \( T(A) \not\subseteq Pol(\rho) \) if \( \rho \) is a relation from Class(4), Class(5), or Class(6) defining maximal clones. By \( T(A) \supseteq Pol(\leq) \) for a connected order \( \leq \) on \( A \), because of Theorem 1.1 we get \( T(A) \not\subseteq Pol(\rho) \) where \( \rho \) is a relation from one of the classes (1) to (4). Then by the Primality Criterion 1.2, \( A \) is primal.

**COROLLARY 4.2.** If \( A = (A; F^A) \) is a finite algebra with \( T(A) \supseteq Pol(\leq) \) for a connected unbounded order \( \leq \) on \( A \), then \( A \) is primal if and only if \( T(A) \not\supseteq Pol(\rho) \) for every relation \( \rho \) from Class(5) and \( V(A) \) is congruence distributive.

**Proof.** \( T(A) \not\subseteq Pol(\rho) \) for a relation \( \rho \) from Class(6). From the assumption and Theorem 1.1 we get primality. The converse is clear.

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