Notes on locally order-polynomially complete lattices

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Dedicated to the memory of Ivan Rival

Let $L = (L, \land, \lor)$ be a lattice. A function $f : L^k \to L$ is called a local polynomial (local term) if for every finite subset $M \subseteq L^k$ there exists a polynomial $p$ (a term function $t$) of $L$ such that $f$ and $p(f$ and $t$) agree on $M$. If every sublattice of $L$ is preserved by $f$ then the function $f$ is called conservative. If all order-preserving functions on $L$ are local polynomials, then $L$ is called locally order-polynomially complete. Locally order-polynomially complete lattices can be characterized in many ways. They are lattices with only trivial tolerances [4, 1]. In this note we analyze the order-preserving conservative functions of a bounded lattice and properties of locally order-polynomially complete lattices. We shall consider lattices $L$ with the property that

1. every order-preserving conservative function of $L$ is a local polynomial.
2. every order-preserving conservative function of $L$ is a local term.

Clearly, any locally order-polynomially complete lattice satisfies (1). A lattice having property (2) will be called locally order-semi-primal.

Lemma 1. Let $(L, \land, \lor)$ be a bounded lattice. Then for any $a \in L$, the following functions $d_a : L^2 \to L$ and $\overline{d_a} : L^2 \to L$ are order-preserving and conservative:

$$d_a(x_1, x_2) = \begin{cases} 1 & \text{for } (x_1, x_2) \geq (1, a) \\ x_1 \land x_2, & \text{else} \end{cases}$$

$$\overline{d_a}(x_1, x_2) = \begin{cases} 0 & \text{for } (x_1, x_2) \leq (0, a) \\ x_1 \lor x_2, & \text{else} \end{cases}$$

Proof. $d_a$ is order-preserving: Take $(x_1, x_2), (y_1, y_2) \in L^2$ with $(x_1, x_2) \leq (y_1, y_2)$. If $d_a(x_1, x_2) \neq 1$ then $d_a(x_1, x_2) \leq d_a(y_1, y_2)$. If $d_a(x_1, x_2) = 1$ then $(y_1, y_2) \geq (1, a)$ implies $d_a(x_1, x_2) = d_a(y_1, y_2) = 1$.

$d_a$ preserves the sublattices of $L$: Let $(S, \land, \lor)$ be a sublattice of $L$ and $c_1, c_2 \in S$. If $(c_1, c_2) \geq (1, a)$ then $c_1 = 1$ implies $1 \in S$ and $d_a(c_1, c_2) = 1$. If $(c_1, c_2) \neq (1, a)$ then we get $d_a(c_1, c_2) = c_1 \land c_2 \in S$. 

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The same assertion can be proved on \( \tilde{d}_a \) by dual reasoning.

A tolerance of a lattice \( L \) is a reflexive, symmetric, and compatible binary relation \( \rho \subseteq L^2 \). \( L \) is tolerance simple (cf. [2]) if it has only the trivial tolerances, namely the identity relation and the all relation.

**Proposition 2.** Let \( L \) be a bounded lattice. If every tolerance of \( L \) is preserved by all binary order-preserving conservative functions of \( L \), then \( L \) is tolerance simple.

**Proof.** Let \( \rho \) be a tolerance of \( L \) different from the identity relation. Then there exist \( a, b \in L \), \( a < b \) with \( (a, b) \in \rho \). As \((0, 0) \in \rho\), \((1, 1) \in \rho\), we obtain \((0, b) = (\tilde{d}_a(0, a), \tilde{d}_a(0, b)) \in \rho\) and \((0, 1) = (\tilde{d}_a(1, 0), \tilde{d}_a(1, b)) \in \rho\). Hence \( \rho \) is the all relation on \( L \).

**Theorem 3.** For any bounded lattice \( L \) the following are equivalent:

(i) \( L \) is locally order-polynomially complete.

(ii) Every order-preserving conservative function of \( L \) is a local polynomial.

(iii) Every binary order-preserving conservative function of \( L \) is a local polynomial.

(iv) \( L \) is tolerance simple.

**Proof.** The implications (i)\(\Rightarrow\)(ii) and (ii)\(\Rightarrow\)(iii) are obvious; (iv)\(\Leftrightarrow\)(i) can be found in [4] or [1].

(iii)\(\Rightarrow\)(iv): Let \( \rho \) be a tolerance of \( L \) and \( f: L^2 \to L \) a binary order-preserving conservative function and let \( (a, b), (c, d) \in \rho \). As by assumption of (iii) there exists a binary polynomial \( p \) of \( L \) such that \( f \) and \( p \) coincide on \( M = \{(a, b), (c, d)\} \subseteq L^2 \), we get \( f((a, b), (c, d)) = p((a, b), (c, d)) \in \rho \). Thus \( \rho \) is preserved by every binary order-preserving conservative function on \( L \) and hence Proposition 2 provides (iv).

We note that the above result can not be sharpened (in spite of the fact that the images of lattice polynomials can be very different, see [3]), as any unary conservative function is the identity on \( L \). Our starting point in characterizing order-semi-primal lattices is the following result of R. Wille:

**Proposition 4 ([5], Corollary 3).** For any finite lattice \( L \) and any integer \( k \geq 1 \) a \( k \)-ary function \( f \) of \( L \) is a term function if and only if

(a) \( f \) is order-preserving and conservative, and

(b) \( \alpha(f(x_1, \ldots, x_k)) \leq f(\alpha(x_1), \ldots, \alpha(x_k)) \) for every \( \vee \)-preserving map \( \alpha \) of \( L \).

**Remark 5.** We note that, even in the case of an infinite \( L \), every term function \( t \) satisfies (b) (as well as (a)). This can be easily proved by induction on “complexity” of \( t \), and is implicitly contained in [5].

**Theorem 6.** Let \( L \) be a bounded lattice. Then the following statements are equivalent:
(i) Every binary order-preserving conservative function of $L$ is a local term function.

(ii) $L$ is locally order-semi-primal.

(iii) $L$ has at most two elements.

Proof. (iii)$\Rightarrow$(ii). It is known that the two element lattice is order-semi-primal. As order-semi-primality implies local order-semi-primality, we are done.

(ii)$\Rightarrow$(i) is straightforward.

(i)$\Rightarrow$(iii): Let $L$ satisfy (i) and suppose $|L| > 2$. Then there is an $a \in L$ with $0 < a < 1$. We define a map $\mu_a$ as follows:

$$\mu_a(x) = 0 \text{ for } x \not\leq a, \text{ and } \mu_a(x) = 1, \text{ otherwise.}$$

Clearly, $\mu_a(x \vee y) = 0 \iff x \vee y \not\leq a \iff x, y \not\leq a \iff \mu_a(x) \vee \mu_a(y) = 0$ and $\mu_a(x) \vee \mu_a(y) = 1 \iff x \vee y \not\geq a \iff \mu_a(x \vee y) = 1$. Thus we get $\mu_a(x \vee y) = \mu_a(x) \vee \mu_a(y)$, for all $x, y \in L$, i.e., $\mu_a$ is $\vee$-preserving.

Now, consider the function $d_a : L^2 \rightarrow L$ defined in Lemma 1. As $d_a$ is order-preserving and conservative, there exists a binary term $t_a$ which coincides with $d_a$ on the set $M = \{(1, a), (\mu_a(1), \mu_a(a))\} \subseteq L^2$. Hence $\mu_a(t_a(1, a)) = \mu_a(d_a(1, a)) = \mu_a(1) = 1$ and $t_a((\mu_a(1), \mu_a(a))) = d_a(\mu_a(1), \mu_a(a)) = d_a(1, 0) = 0$. Thus we obtain $\mu_a(t_a(1, a)) \notin t_a((\mu_a(1), \mu_a(a)))$, a contradiction in view of [5] and Remark 5.  

REFERENCES


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