ENOMORPHISMS OF QUASIORDERS
AND RELATED LATTICES

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Abstract. In this paper we investigate lattices of binary relations \( \sigma \subseteq A \times A \) preserved by the endomorphisms or polymorphisms of a quasiorder \( q \subseteq A \times A \), i.e. with the property \( \text{End } q \subseteq \text{End } \sigma \). In particular we determine the quasiorder lattice of the algebra \((A, \text{Pol}_q)\) and derive conditions for tolerance and congruence simplicity.

Introduction

The quasiorders (i.e. compatible reflexive and transitive relations) of an algebra can be considered as a common generalization of its congruences and compatible partial orders. Their study has a particular importance in the case of partially ordered algebras, see e.g. [CzéL83b], [CzéL83a], [ChaPD99] or [RadS05]. For instance, any so-called order-congruence (see [CzéL83a]) of a partially ordered algebra \((A, F)\) has the form \( q \cap q^{-1} \), where \( q \) is a quasiorder of \((A, F)\).

The lattice of all quasiorders on a given set has several special properties (e.g. it is atomistic, dually atomistic and complemented, [ErnR95]). The situation changes if we consider quasiorder lattices of algebras: every algebraic lattice is isomorphic to the quasiorder lattice of a suitable algebra (see e.g. [ChaC96], [Pin95]). Only few properties of the lattices of quasiorders are known in the case of concrete classes of algebras (e.g. varieties). For majority algebras it is known that their quasiorder lattices are always distributive (see [CzéL83a] or [PinC93]).

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One of the starting points for this paper was the question how endomorphisms of quasiorders behave, in particular under which conditions we have \( \text{End}\ q \subseteq \text{End}\ q' \) for quasiorders \( q, q' \). This question is equivalent (see 1.1) to the question which \( q' \) belong to the quasiorder lattice \( \text{Quord}(A, \text{End}\ q) \).

We investigate the lattice of invariant (compatible) binary relations of the algebras \((A, \text{End}\ q)\) and \((A, \text{Pol}\ q)\) (\( \text{End}\ q \) and \( \text{Pol}\ q \) denote the endomorphism monoid and the clone of polymorphisms of a quasiorder \( q \), see 1.1; contrary to the above mentioned examples, \((A, \text{Pol}\ q)\) is not a majority algebra in general). The structure of these lattices is given in Theorem 2.3. As a corollary we get that the quasiorders of the algebra \((A, \text{Pol}\ q)\) form a particular (very small) distributive lattice (Corollary 2.5).

We also collect several easy consequences concerning congruence or tolerance simplicity of the algebras under consideration. Moreover, we show that \( \text{End}\ q \) is a dual atom in the poset \( \{\{\text{End}\ \varrho \mid \varrho \subseteq A \times A\}, \subseteq\}\) if and only if either \( q \) is a linear order or a nontrivial equivalence. This extends some results of \([\text{GibZ98, Thm. 1}]\) and \([\text{PecM05}]\) dealing with the question when \( \text{Pol}\ \varrho \subseteq \text{Pol}\ \sigma \) or \( \text{End}\ \varrho \subseteq \text{End}\ \sigma \) for partial orders or equivalence relations \( \varrho, \sigma \).

We focus here mainly on quasiorders \( q \) but a detailed study of the proofs shows that several results can be generalized to reflexive relations \( q \) not necessarily transitive.

1. Preliminaries

1.1. Operations and relations. Let \( \text{Op}(A) \) (\( \text{Op}^{(1)}(A) \), resp.) and \( \text{Rel}^{(2)}(A) \) denote the set of all finitary (unary, resp.) operations and the set of all binary relations on a set \( A \). The equivalence relations \( \Delta := \{(a, a) \mid a \in A\} \) and \( \nabla := A \times A \) are called trivial. An \( n \)-ary operation \( f : A^n \to A \) preserves a binary relation \( \varrho \subseteq A \times A \) (or \( \varrho \) is invariant for, or compatible with \( f \)) if

\[
(a_1, b_1), \ldots, (a_n, b_n) \in \varrho \implies (f(a_1, \ldots, a_n), f(b_1, \ldots, b_n)) \in \varrho
\]

for all \( a_1, \ldots, a_n, b_1, \ldots, b_n \in A \). We use the notation

\[
f \triangleright \varrho \quad \text{or} \quad F \triangleright \varrho
\]

if \( f \) preserves \( \varrho \), or if every \( f \in F \subseteq \text{Op}(A) \) preserves \( \varrho \).

For \( \varrho \in \text{Rel}^{(2)}(A) \) and an algebra \( (A, F) \), where \( F \subseteq \text{Op}(A) \), we introduce the notation

\[
\begin{align*}
\text{Pol}\ \varrho &:= \{f \in \text{Op}(A) \mid f \triangleright \varrho\}, \\
\text{End}\ \varrho &:= \{f \in \text{Op}^{(1)}(A) \mid f \triangleright \varrho\}, \\
\text{Inv}^{(2)}(A, F) &:= \text{Inv}^{(2)} F := \{\varrho \in \text{Rel}^{(2)}(A) \mid \forall f \in F : f \triangleright \varrho\}, \\
\text{Con}(A, F) &:= \{\varrho \in \text{Inv}^{(2)}(A, F) \mid \varrho \text{ is an equivalence relation}\}, \\
\text{Tol}(A, F) &:= \{\varrho \in \text{Inv}^{(2)}(A, F) \mid \varrho \text{ is a tolerance relation}\},
\end{align*}
\]
Quord(\(A, F\)) := \{\varrho \in \text{Inv}^{(2)}(A, F) \mid \varrho \text{ is a quasiorder}\} (see 1.2, 1.3).

There are several Galois connections induced by \(\triangleright\), which were intensively studied (e.g. Pol−Inv, Pol−Inv\(^{(2)}\), End−Inv, for details see e.g. [PösK79], [Pös04]). Pol\(\varrho\) is always a clone, End\(\varrho\) is the endomorphism monoid of \(\varrho\), Con(\(A, F\)) and Tol(\(A, F\)) is the congruence and tolerance lattice, respectively, of the algebra \((A, F)\). Moreover, Inv\(^{(2)}\)(\(A, F\)) equals (with possible exception of the empty set) the subalgebra lattice of the direct square \((A, F)^2\) of the algebra \((A, F)\). The subalgebra generated by a set \(\sigma \subseteq A \times A\) (i.e. the least invariant relation containing \(\sigma\)) will be denoted by \(\langle \sigma \rangle\) or \(\langle \sigma \rangle_{(A,F)}\).

1.2. Quasiorders. A quasiorder \(q\) on a set \(A\) is a reflexive and transitive relation \(q \subseteq A \times A\). The set of all quasiorders on \(A\) is denoted by Quord(\(A\)). For \(q \in \text{Quord}(A)\) the inverse \(q^{-1} := \{(y, x) \mid (x, y) \in q\}\) is also a quasiorder and the relation \(q_0 := q \cap q^{-1}\) is an equivalence on \(A\). Obviously, a quasiorder \(q\) is a partial order if and only if \(q_0 = q \cap q^{-1} = \Delta\).

A binary relation \(\varrho \subseteq A \times A\) is called connected\(^1\) if \((\varrho \cup \varrho^{-1})^{(\triangledown)} = \nabla\) (here \(\sigma^{(\triangledown)}\) denotes the transitive closure of a binary relation \(\sigma\)). Further, a partial order \(q\) is called locally bounded if any two elements \(a, b \in A\) have an upper bound and a lower bound in the poset \((A, q)\).

Quord(\(A\)) forms a complete lattice with respect to inclusion \(\subseteq\). Meet and join are given by

\[\bigwedge\{q_i \mid i \in I\} = \bigcap\{q_i \mid i \in I\}\] and \[\bigvee\{q_i \mid i \in I\} = (\bigcup\{q_i \mid i \in I\})^{(\triangledown)}\]

for any \(q_i \in \text{Quord}(A)\), \(i \in I\), and least and greatest elements are \(\Delta\) and \(\nabla\), respectively. The mapping \(q \mapsto q^{-1}\) is an involution of the lattice \((\text{Quord}(A), \cap, \lor)\), whose fixed points \(q = q^{-1}\) are exactly the equivalence relations on \(A\).

1.3. Quasiorders of algebras. The quasiorders \(q \in \text{Quord}(A, F)\) of an algebra \((A, F)\) (i.e. the quasiorders compatible with \(F\), cf. 1.1) form a complete sublattice of \((\text{Quord}(A), \subseteq)\) (with meet and join as given in 1.2). The fixed points \(q = q^{-1}\) of the involution \(q \mapsto q^{-1}\) are exactly the congruences of the algebra \((A, F)\).

\(^1\)i.e. the graph with vertex set \(A\) and edge set \(\varrho\) is connected in the usual graph theoretic sense.
Let $P^{(1)}(F)$ denote the set of all unary polynomials of the algebra $(A, F)$ (i.e. the unary part of the clone generated by $F$ and by all constant functions). It is well-known (see e.g. [RadS05, p. 44]) that

\begin{equation}
\text{Quord}(A, F) = \text{Quord}(A, P^{(1)}(F)).
\end{equation}

(1.3.1)

For a quasiorder $q \in \text{Quord}(A)$ (or more generally for any reflexive relation) we have $P^{(1)}(\text{Pol} q) = \text{End} q$ (since every constant mapping preserves $q$), thus the above equation implies in particular

\begin{equation}
\text{Quord}(A, \text{Pol} q) = \text{Quord}(A, \text{End} q).
\end{equation}

(1.3.2)

However, note that in general $\text{Inv}^{(2)}(A, \text{Pol} q) \neq \text{Inv}^{(2)}(A, \text{End} q)$. Obviously every invariant relation is reflexive.

1.4. Endomorphisms. By definition, a mapping $f : A \to A$ is an endomorphism of a relation $\varrho \in \text{Rel}^{(2)}(A)$ if and only if $f \triangleright \varrho$, i.e. $(f(x), f(y)) \in \varrho$ for all $(x, y) \in \varrho$. To give an example let $(a, a), (a, b), (b, a), (b, b) \in \sigma \subseteq A \times A$, $a_0 \in A$ and let $f : A \to A$ be defined by

$$f(x) := \begin{cases} 
b & \text{if } x = a_0, \\
a & \text{otherwise.}\end{cases}$$

Then $f \triangleright \sigma$. In fact, trivially we have $(f(x), f(y)) \in \{(a, a), (a, b), (b, a), (b, b)\} \subseteq \sigma$ for arbitrary $x, y \in A$, in particular for $(x, y) \in \sigma$.

1.5. The lattice $\text{Inv}^{(2)}(A, \text{End} q)$. Since we want to investigate the lattice $L' := (\text{Inv}^{(2)}(A, \text{End} q), \cap, \cup)$ in more detail we mention here some properties which directly follow from well-known facts. $L'$ is completely distributive since meet and join are the set-theoretical intersection and union. Moreover, because $L'$ is the subalgebra lattice of a unary algebra (namely the direct square of $(A, \text{End} q)$), $L'$ is CJ-generated, i.e. every element is the union of completely join-irreducible elements.

2. The binary invariant relations of $(A, \text{End} q)$ and $(A, \text{Pol} q)$

In this section we shall describe the structure of the lattice $\text{Inv}^{(2)}(A, \text{End} q)$ of invariant relations. After this, some of the consequences of the theorem (which might be of its own interest, too) are given, in particular one gets a full description of the quasiorder lattice $\text{Quord}(A, F)$ for algebras with $F = \text{Pol} q$, where $q$ is a quasiorder (Corollary 2.5).
2.1. As preparation for Theorem 2.3 we introduce some special relations derived from a given quasiorder \( q \in \text{Quord}(A) \). For \( k \in \mathbb{N} \), let

\[
\varepsilon_k := \underbrace{q \circ q^{-1} \circ \ldots \circ q^i}_{k \text{ factors}} \text{ with } \alpha = 1 \text{ for odd } k \text{ and } \alpha = -1 \text{ for even } k,
\]

\[
\varepsilon_{-k} := \underbrace{q^{-1} \circ q \circ \ldots \circ q^i}_{k \text{ factors}} \text{ with } \beta = -1 \text{ for odd } k \text{ and } \beta = 1 \text{ for even } k.
\]

In particular, \( \varepsilon_1 = q \), \( \varepsilon_{-1} = q^{-1} \), \( \varepsilon_k^{-1} = \varepsilon_k \), \( \varepsilon_{-k}^{-1} = \varepsilon_{-k} \) for even \( k \), \( \varepsilon_k^{-1} = \varepsilon_{-k} \) for odd \( k \), and the properties of \( q \) imply

\[
\varepsilon_k \cup \varepsilon_{-k} \subseteq \varepsilon_{k+1} \cap \varepsilon_{-(k+1)}, \quad q \cup q^{-1} = \bigcup_{k \in \mathbb{N}} \varepsilon_k = \bigcup_{k \in \mathbb{N}} \varepsilon_{-k},
\]

see also Fig. 2.3.1(\( L' \)).

The following lemma is crucial for the proof of the next theorem.

**Lemma 2.2.** Let \( q \in \text{Quord}(A) \) and \( \sigma \in \text{Inv}^{(2)}(A, \text{End} q) \). Then, for \( k \geq 1 \), we have \( \sigma \not\subseteq \varepsilon_{-k} \implies \varepsilon_k \subseteq \sigma \) and dually \( \sigma \not\subseteq \varepsilon_k \implies \varepsilon_{-k} \subseteq \sigma \).

**Proof.** For \( \varepsilon_k = q \circ q^{-1} \circ \ldots \circ q^i \) (see 2.1) we need also reverse products of the factors. Therefore we introduce the following notation:

\[
\varepsilon_k := q^{\alpha_1} \circ q^{\alpha_2} \circ \ldots \circ q^{\alpha_k}, \quad \text{i.e. } (\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_k) := (1, -1, 1, \ldots, \alpha),
\]

\[
\delta_i := q^{\alpha_k} \circ q^{\alpha_{k-1}} \circ \ldots \circ q^{\alpha_i} \text{ for } i \in \{1, \ldots, k\}.
\]

Clearly (recall that \( q \) is transitive and reflexive) we have

\[
\delta_1 \supseteq \delta_2 \supseteq \ldots \supseteq \delta_k, \quad \delta_i \circ q = \begin{cases} 
\delta_{i-1} & \text{if } \alpha_i = 1 \ (i \in \{1, \ldots, k\}), \\
\delta_i & \text{if } \alpha_i = -1 \ (i \in \{2, \ldots, k\}).
\end{cases}
\]

In order to prove the Lemma, assume \( \sigma \not\subseteq \varepsilon_{-k} \), i.e., there exists \( (a_0, b_0) \in \sigma \) with \( (a_0, b_0) \notin \varepsilon_{-k} \) (note that this is equivalent to \( (b_0, a_0) \notin \delta_1 \)). Further let \( (a, b) \in \varepsilon_k \) (we have to show \( (a, b) \in \sigma \)). By definition of \( \varepsilon_k \) there exist \( c_0, \ldots, c_k \in A \) such that \( a = c_0, b = c_k \) and \( (c_i, c_{i+1}) \in q^{\alpha_{i+1}} \) for \( i \in \{0, 1, \ldots, k-1\} \). We define a function \( f : A \to A \) as follows:

\[
f(x) := \begin{cases} 
c_k (= b) & \text{if } (b_0, x) \in \delta_k (= \delta_k \setminus \emptyset), \\
\vdots & \\
c_i & \text{if } (b_0, x) \in \delta_i \setminus \delta_{i+1}, \\
\vdots & \\
c_1 & \text{if } (b_0, x) \in \delta_1 \setminus \delta_2, \\
c_0 (= a) & \text{otherwise, i.e., if } (b_0, x) \in \nabla \setminus \delta_1.
\end{cases}
\]
We have \( f(a_0) = a \) (since \((b_0, a_0) \notin \delta_1\)) and \( f(b_0) = b \) (since \((b_0, b_0) \in \delta_k\)). We shall prove \( f \circ q \) (see the claim below), therefore we have \( f \in \text{End } q \subseteq \text{End } \sigma \), and \((a_0, b_0) \in \sigma\) implies \((a, b) = (f(a_0), f(b_0)) \in \sigma\) what finishes the proof.

Claim: \( f \circ q \).

Let \((x, y) \in q\) (we have to show \((f(x), f(y)) \in q\)). Let \(i \in \{1, \ldots, k\}\) and assume \(f(x) = c_i, \text{i.e., } (b_0, x) \in \delta_i \setminus \delta_{i+1}\) (formally here we may use \(\delta_0 := \nabla\) and \(\delta_{k+1} := \emptyset\)). Since \((x, y) \in q\) we have \((b_0, y) \in \delta_i \circ q\).

**Case 1:** \(\alpha_i = 1\). Then \(\delta_i \circ q = \delta_i\) and \((b_0, y) \in \delta_i\). For \(i = k\) this implies \(f(y) = c_i\). Otherwise \((i < k)\) we have \((b_0, y) \notin \delta_{i+1}\) since \((b_0, y) \in \delta_{i+1} = q^{a_k} \circ \ldots \circ q^{a_1}\) (note \(\alpha_{i+1} = -1\)) together with \((y, x) \in q^{-1}\) would imply \((b_0, x) \in \delta_{i+1}\), a contradiction. Thus also \(f(y) = c_i\) and \((f(x), f(y)) = (c_i, c_i) \in q\).

**Case 2:** \(\alpha_i = -1\). Since \(\alpha_1 = 1\), this can occur only for \(i > 1\). Then \(\delta_i \circ q = \delta_{i-1}\) and \((b_0, y) \in \delta_{i-1}\). But \((b_0, y) \notin \delta_{i+2}\): for \(i = k - 1\) this is trivial (note \(\delta_{k+1} = \emptyset\), for \(i = k\) see remarks below, so we may assume \(i < k - 1\)) and then \((b_0, y) \in \delta_{i+2} = q^{a_k} \circ \ldots \circ q^{-1}\) together with \((y, x) \in q^{-1}\) would imply \((b_0, x) \in \delta_{i+2} \circ q^{-1} = \delta_{i+2} \subseteq \delta_{i+1}\), a contradiction. Therefore either \((b_0, y) \in \delta_{i-1} \setminus \delta_i\) or \((b_0, y) \in \delta_i \setminus \delta_{i+1}\) or \((b_0, y) \in \delta_{i+1} \setminus \delta_{i+2}\) (for \(i = k\) only the first two possibilities can occur, since \((b_0, y) \notin \delta_{i+1}\) trivially) showing \(f(y) \in \{c_i, c_{i-1}, c_{i+1}\}\) and we have \((f(x), f(y)) \in \{(c_i, c_{i-1}), (c_i, c_i), (c_i, c_{i+1})\} \subseteq (q^{a_i})^{-1} \cup q^{a_i+1} = q \cup q = q\) (for \(i = k\) we get only \(f(y) \in \{c_{i-1}, c_i\}\)).

Finally, we prove the remaining case \(i = 0\): Let \(f(x) = c_0\). Then \((b_0, x) \notin \delta_1 = q^{a_k} \circ \ldots \circ q^{a_1}\) (\(\alpha_1 = 1\)). Moreover, \((b_0, y) \notin \delta_2\) since \((b_0, y) \in \delta_2 = q^{a_k} \circ \ldots \circ q^{-1}\) together with \((y, x) \in q^{-1}\) would imply \((b_0, x) \in \delta_2 \subseteq \delta_1\), a contradiction. Thus \(f(y) \in \{c_0, c_1\}\) and \((f(x), f(y)) \in \{(c_0, c_0), (c_0, c_1)\} \subseteq q\).

This finishes the proof of the first implication of the lemma. Taking into account \(\text{End } q = \text{End } q^{-1}\), the second implication follows from the first one by changing the role of \(q\) and \(q^{-1}\).

The following Theorem is visualized in Fig. 2.3.1(\(L'\)) and (\(L\)). In these figures, straight lines mean that there are no invariant relations in between, while the shaded parts in general are nonempty sublattices.

**Theorem 2.3.** Let \(q \in \text{Quord}(A)\).

(A) If \(\sigma \in \text{Inv}^{(2)}(A, \text{End } q)\) then one of the following conditions is fulfilled:

(i) \(\sigma \in \{\Delta, q \lor q^{-1}, \nabla\}\),

(ii) there exists \(k \geq 1\) such that \(\sigma \in \{\varepsilon_k \cap \varepsilon_\neg k, \varepsilon_k, \varepsilon_\neg k, \varepsilon_k \cup \varepsilon_\neg k\}\),

(iii) there exists \(k \geq 2\) such that \(\varepsilon_k \cup \varepsilon_\neg k \subseteq \sigma \subseteq \varepsilon_k + 1 \cap \varepsilon_\neg (k+1)\).

(B) If \(\sigma \in \text{Inv}^{(2)}(A, \text{Pol } q)\) then one of the following conditions is fulfilled:

(i) \(\sigma \in \{\Delta, q \lor q^{-1}, \nabla\}\),

(ii) there exists \(k \geq 1\) such that \(\sigma \in \{\varepsilon_k \cap \varepsilon_\neg k, \varepsilon_k, \varepsilon_\neg k, \langle \varepsilon_k \cup \varepsilon_\neg k \rangle_{(A, \text{Pol } q)}\}\),

(iii) there exists \(k \geq 2\) such that \(\langle \varepsilon_k \cup \varepsilon_\neg k \rangle_{(A, \text{Pol } q)} \subseteq \sigma \subseteq \varepsilon_k + 1 \cap \varepsilon_\neg (k+1)\).
Figure 2.3.1. The structure of $L' = \text{Inv}^{(2)}(A, \text{End} q)$ and $L = \text{Inv}^{(2)}(A, \text{Pol} q)$

Proof. It is clear from the definitions that all relations $\sigma$ in (i) and (ii) are invariant for $\text{End} q$ or $\text{Pol} q$, respectively. This follows from the well-known fact (see e.g. [Pösk79]), that $\text{Inv} \text{End} q$ ($\text{Inv} \text{Pol} q$, resp.) is closed under constructions using positive (primitive positive, resp.) first order $\exists$-formulas, in
Claim 1: \( q \cup q^{-1} = \varepsilon_1 \cup \varepsilon_{-1} \subseteq \varepsilon_2 \cap \varepsilon_{-2} \).

Let \( q \cup q^{-1} = \varepsilon_1 \cup \varepsilon_{-1} \subseteq \varepsilon_2 \cap \varepsilon_{-2} \). Then there exists \( (x_0, y_0) \in \sigma \subseteq \varepsilon_2 \cap \varepsilon_{-2} \) with \( (x_0, y_0) \notin q \cup q^{-1} \). Further let \( (a, b) \in (\varepsilon_2 \cap \varepsilon_{-2}) \setminus (q \cup q^{-1}) \). By definition of \( \varepsilon_2 = q \circ q^{-1} \) and \( \varepsilon_{-2} = q^{-1} \circ q \), there exist \( c, d \in A \) such that

\[ (a, c) \in q, \ (c, b) \in q^{-1} \text{ and } (a, d) \in q^{-1}, \ (d, b) \in q. \]
Consider the mapping \( f : A \to A \) defined by

\[
f(x) := \begin{cases} 
  d & \text{if } (x,x_0) \in q \text{ and } (x,y_0) \in q, \\
  a & \text{if } (x,x_0) \in q \text{ and } (x,y_0) \notin q, \\
  b & \text{if } (x,x_0) \notin q \text{ and } (x,y_0) \in q, \\
  c & \text{otherwise.}
\end{cases}
\]

We prove \( f \triangleright q \) (and therefore also \( f \triangleright \sigma \)). In fact, let \((x,y) \in q\).

If \( f(y) = c \), then \((f(x), f(y)) \in \{(d,c), (a,c), (b,c), (c,c)\} \subseteq q\).

If \( f(y) = a \), then \((y,x_0) \in q \) and therefore \((x,x_0) \in q \circ q = q\). Thus \( f(x) \in \{d,a\}, \) i.e. \((f(x), f(y)) \in \{(d,a), (a,a)\} \subseteq q\). Analogously one proves \((f(x), f(y)) \in q \) in case \( f(y) = b \).

Finally, if \( f(y) = d \), then \((y,x_0) \in q\), \((y,y_0) \in q\), thus (together with \((x,y) \in q\)) we get \((x,x_0) \in q\), \((x,y_0) \in q\), i.e. \( f(x) = d \). Consequently \((f(x), f(y)) = (d,d) \in q\), and \( f \triangleright q \) is proved.

Now, by definition \( f(x_0) = a\), \( f(y_0) = b\). Because \( f \triangleright \sigma \) we get \((a,b) = (f(x_0), f(y_0)) \in \sigma\). Since \((a,b)\) was chosen arbitrarily, we get \( \varepsilon_2 \cup \varepsilon_{-2} \subseteq \sigma\), i.e. \( \sigma = \varepsilon_2 \cup \varepsilon_{-2}\).

Summarizing, for any relation \( \sigma \in \text{Inv}^{(2)} \text{End } q \) one of the conditions (i), (ii) or (iii) of part (A) of the Theorem is fulfilled.

(B): Because \( \text{Inv}^{(2)} \text{Pol } q \subseteq \text{Inv}^{(2)} \text{End } q \), (B) immediately follows from (A) (note that \( \varepsilon_k \cup \varepsilon_{-k} \subseteq \sigma \in \text{Inv}^{(2)} \text{Pol } q \) implies \( \langle \varepsilon_k \cup \varepsilon_{-k} \rangle \langle A, \text{Pol } q \rangle \subseteq \sigma \)), however with one exception: it remains to prove the following

Claim 2: \( \langle \varepsilon_1 \cup \varepsilon_{-1} \rangle = \varepsilon_2 \cap \varepsilon_{-2}\).

If \( q = q^{-1} \) then \( q = \varepsilon_k = \varepsilon_{-k} \) for all \( k \) and the statement is trivial. Thus we may assume that there exists \((a_0, b_0) \in q \setminus q^{-1}\).

We are going to show \((a,b) \in \langle q \cup q^{-1} \rangle \langle A, \text{Pol } q \rangle \) for any given \((a,b) \in \langle \varepsilon_2 \cap \varepsilon_{-2} \rangle \setminus \langle q \cup q^{-1} \rangle \) by constructing a binary function \( q \in \text{Pol } q \) such that \((a,b) = (g(a_0, b_0), g(b_0, a_0))\) (note that \( \{(a_0, b_0), (b_0, a_0)\} \subseteq q \cup q^{-1}\)).

By definition of \( \varepsilon_2, \varepsilon_{-2} \) there exist \( c, d \in A \) (as in the proof of Claim 1 above) such that \((a,c) \in q\), \((c,b) \in q^{-1}\), \((a,d) \in q^{-1}\), \((d,b) \in q\). For abbreviation we introduce

\[
\tilde{q} := \{(x,y), (x', y')\} \mid (x,x') \in q, (y,y') \in q\}.
\]

Let \( g : A \times A \to A \) be defined by

\[
g(x_1, x_2) := \begin{cases} 
  d & \text{if } ((x_1, x_2), (a_0, b_0)) \in \tilde{q} \text{ and } ((x_1, x_2), (b_0, a_0)) \in \tilde{q}, \\
  a & \text{if } ((x_1, x_2), (a_0, b_0)) \in \tilde{q} \text{ and } ((x_1, x_2), (b_0, a_0)) \notin \tilde{q}, \\
  b & \text{if } ((x_1, x_2), (a_0, b_0)) \notin \tilde{q} \text{ and } ((x_1, x_2), (b_0, a_0)) \in \tilde{q}, \\
  c & \text{otherwise.}
\end{cases}
\]

Then \( g \triangleright q \) (cf. 1.1) is equivalent to the implication \(((x_1, y_1), (x_2, y_2)) \in \tilde{q} \implies \langle g(x_1, x_2), g(y_1, y_2) \rangle \in q \) for all \((x_i, y_i) \in A\). This can be proved in full analogy to the proof \( f \triangleright q \) given for Claim 1 above (with \( x, y, x_0, y_0, q \) replaced by
\[(x_1, x_2), (y_1, y_2), (a_0, b_0), (b_0, a_0), \tilde{q}, \text{ respectively}, \) so it will not be done here (thinking of \(q\) as a partial order, the representation of \(g\) in Fig. 2.3.2 may provide some insight, why \(g\) maps elements of \(\tilde{q}\) to elements of \(q\)).

As already mentioned \(g \triangleright q\) implies \((a, b) \in \{q \cup q^{-1}\}_{(A, \text{Pol} q)}\), consequently \((q \cup q^{-1})_{(A, \text{Pol} q)} = \varepsilon_2 \cap \varepsilon_{-2} \).

\[
A \times A \xrightarrow{g} A
\]

**Figure 2.3.2.** The mapping \(g\) preserves \(q\)

**Remarks 2.4.**

1. Fig. 2.3.1 represents the most general situation. For special \(q\) several parts may collapse (see 2.5). For example, we may have \(\varepsilon_k \subseteq \varepsilon_{-k}\) (or \(\varepsilon_{-k} \subseteq \varepsilon_k\)). Then, by easy computing, we obtain \(\varepsilon_{-k} = q \lor q^{-1}\) (or \(\varepsilon_k = q \lor q^{-1}\)), moreover, \(\varepsilon_{-k}\) covers \(\varepsilon_k\) (or \(\varepsilon_k\) covers \(\varepsilon_{-k}\)) if \(k\) is even, and \(\varepsilon_k = \varepsilon_{-k}\) if \(k\) is odd. In general, the shaded intervals have their own special structure which would lead far beyond the scope of our aims here and would deserve another paper. Here we only show that these intervals are nonempty in general. We give an example: Fig. 2.4.1 shows the (Hasse-)diagram of a partial order \(q\) on a 13-element set. Let

\[
\tau := \{(x, y) \mid \exists w : x \varepsilon_2 w \varepsilon_2 y \& x \varepsilon_{-2} w \varepsilon_{-2} y\},
\]
i.e. \(\tau\) contains all pairs \((x, y)\) such that there exist elements \(w\) and \(z_1, z_2, u_1, u_2\) which are related to each other as indicated in parentheses in the first part of Fig. 2.4.1. By construction (primitive positive first order formula) we have
But in which are always true in the following general situation: Let $A$ algebra and e.g. the relations nevertheless there are some connections between the join-irreducible elements, $\tau \in \text{Inv}^{(2)} \text{Pol} q$. It is easy to check that $e_2 \cup e_{-2} \subseteq \tau \subseteq e_3 \cap e_{-3}$, $(a, b) \in \tau \setminus (e_2 \cup e_{-2})$, but $(a', b') \in (e_3 \cap e_{-3}) \setminus \tau$. This implies $e_2 \cup e_{-2} \nsubseteq \tau \subseteq e_3 \cap e_{-3}$. Moreover, $(a, b) \notin (e_2 \cup e_{-2})(A, \text{Pol} q)$ (thus we also have $(e_2 \cup e_{-2}) \nsubseteq \tau$), since otherwise there would exist a function $f \in \text{Pol} q$ such that

$$
\begin{pmatrix}
(a) \\
(b)
\end{pmatrix} = \begin{pmatrix}
f(r_1(1), \ldots, r_m(1), s_1(1), \ldots, s_n(1)) \\
f(r_1(2), \ldots, r_m(2), s_1(2), \ldots, s_n(2))
\end{pmatrix}
$$

with $r_i := (r_i(1)) \in q$ and $s_j := (s_j(1)) \in q^{-1}$ for $i \in \{1, \ldots, m\}$, $j \in \{1, \ldots, n\}$. Let $c := f(r_1(2), \ldots, r_m(2), s_1(1), \ldots, s_n(1))$. Because $f \triangleright q$ and $(r_i(1)) \in q$ and $(s_j(2)) \in q$ (and because $q$ is reflexive), we get $(a, c) \in q$ as well as $(b, c) \in q$. But in $q$ there is no such common upper bound $c$ of $a$ and $b$, a contradiction.

(2) The lattice $L' := (\text{Inv}^{(2)} \text{End} q, \subseteq)$ is distributive and CJ-generated (cf. 1.5). Nothing similar is known about the lattice $L := (\text{Inv}^{(2)} \text{Pol} q, \subseteq)$, nevertheless there are some connections between the join-irreducible elements, e.g. the relations $e_k, e_{-k}$ are completely join-irreducible (as well as completely meet-irreducible) in both lattices (see Fig. 2.3.1). It is not hard to check that these relations are neutral in $L$ (i.e. the sublattice of $L$ generated by $e_k$ (or $e_{-k}$, resp.) and any two $x, y \in L$ is distributive). Moreover we can exploit results which are always true in the following general situation: Let $A = (A, F')$ be an algebra and $A' := (A, \langle F' \rangle^{(1)})$ its unary reduct (where $\langle F' \rangle^{(1)}$ denotes the set of all unary term functions of $A$), further let $L$ and $L'$ be the subalgebra lattices of $A$ and $A'$, resp., and let $\text{CJ}(L)$ denote the set of completely join-irreducible elements of a lattice $L$. Then we have:

$$(*) \quad \text{CJ}(L) \subseteq \text{CJ}(L') \subseteq L \subseteq L'.$$

This can be easily derived from the facts, that $L'$ is CJ-generated with meet and join being set-theoretical intersection and union respectively, every completely join-irreducible element $B$ of a subalgebra lattice is 1-generated (note that every $B \in L$ is the join of all $\langle b \rangle_A$ with $b \in B$), and the 1-generated subalgebras of $A$ and $A'$ coincide, i.e. $\langle b \rangle_A = \langle b \rangle_{A'}$. 

\[\begin{array}{c}
\text{Figure 2.4.1. A quasiorder } q \text{ with } \langle e_2 \cup e_{-2} \rangle \nsubseteq \tau \nsubseteq e_3 \cap e_{-3} \\
\end{array} \]
Applying (*) to the above lattices $L$ and $L'$ of binary invariant relations of $\text{Pol}_q$ and $\text{End}_q$ we get

\[ \text{CJ}(\text{Inv}^{(2)} \text{Pol}_q) \subseteq \text{CJ}(\text{Inv}^{(2)} \text{End}_q) \subseteq \text{Inv}^{(2)} \text{Pol}_q \subseteq \text{Inv}^{(2)} \text{End}_q. \]

In particular this implies that every $\sigma \in L = \text{Inv}^{(2)} \text{Pol}_q$ is the union of relations which are both, invariant for $\text{Pol}_q$ and completely join-irreducible in the lattice $L' = \text{Inv}^{(2)} \text{End}_q$.

From Theorem 2.3 many results can be derived more or less easily. The following corollary in particular answers the question about the quasiorder lattices $\text{Quord}(A, \text{Pol}_q) = \text{Quord}(A, \text{End}_q)$ of algebras of the form $(A, \text{Pol}_q)$.

**Corollary 2.5.** Let $q \in \text{Quord}(A)$. Then the quasiorder lattice $\text{Quord}(A, \text{End}_q)$ is a distributive lattice with at most six elements (cf. Fig. 2.5.1(a)). In particular we have:

(i) $\text{Quord}(A, \text{End}_q) = \{\Delta, q_0, q, q^{-1}, q \lor q^{-1}, \nabla\}$.

(ii) If $q$ is a partial order on $A$ then

$\text{Quord}(A, \text{End}_q) = \{\Delta, q, q^{-1}, q \lor q^{-1}, \nabla\}$.

(iii) If $q$ is a connected partial order on $A$ then (cf. Fig. 2.5.1(b))

$\text{Quord}(A, \text{End}_q) = \{\Delta, q, q^{-1}, \nabla\}$.

(iv) If $q$ is a linear order on $A$ then

$\text{Quord}(A, \text{End}_q) = \text{Inv}^{(2)}(A, \text{End}_q) = \{\Delta, q, q^{-1}, \nabla\}$.

(v) $q$ is an equivalence relation on $A$ if and only if (cf. Fig. 2.5.1(c))

$\text{Quord}(A, \text{End}_q) = \text{Inv}^{(2)}(A, \text{End}_q) = \{\Delta, \nabla\}$.

**Proof.** (i): The inclusion \(\supseteq\) is clear since all indicated relations belong to the quasiorder lattice generated by $q$ (cf. 1.2). The opposite inclusion \(\subseteq\) follows from 2.3(A), since for a quasiorder $\sigma$ the inclusion $q \cup q^{-1} \subseteq \sigma$ implies...
proof. (1) Let $(\varepsilon_{\text{a maximal locally closed clone; see \cite[Pösk79, 4.3.7]{Pösk79}}} )$ in case of finite (cf. 1.2). The “if”-part of (2) is a known fact (e.g. for proving that $\text{Pol}$ Corollary 2.6. Let $(\varepsilon_{\text{a lower bound and an upper bound, i.e.}} )$ and only if either $q_0 = \Delta$ for any partial order $q$, and $q \vee q^{-1} = (q \cup q^{-1})^\circ = \nabla$ for a connected partial order (see 1.2). If $q$ is a linear order then $q \vee q^{-1} = q \cup q^{-1} = \nabla$, hence (iv) follows from Theorem 2.3(A). Finally, $q$ is an equivalence if and only if $q_0 = q = q^{-1} = q \cup q^{-1} = q \vee q^{-1}$, hence (v) also follows from 2.3(A). \hfill \Box

We recall the following notions. A tolerance relation is a reflexive and symmetric binary relation. According to 1.1, $\text{Tol}(A,F)$ denotes the set of all tolerances preserved by $F$. An algebra is called tolerance simple (congruence simple, resp.) if it has only trivial tolerances (congruences, resp.), i.e. if $\text{Tol}(A,F) = \{\Delta, \nabla\}$ (Con$(A,F) = \{\Delta, \nabla\}$, resp.).

Corollary 2.6. Let $q \in \text{Quord}(A) \setminus \{\Delta, \nabla\}$.

1. $(A,\text{Pol} q)$ (or equivalently $(A, \text{End} q)$) is congruence simple if and only if $q$ is a connected partial order.
2. $(A,\text{Pol} q)$ is tolerance simple if and only if $q$ is a locally bounded partial order.
3. $(A, \text{End} q)$ is tolerance simple if and only if $q$ is a linear order.

Proof. (1) Let $(A,\text{Pol} q)$ be congruence simple. Since $q \cap q^{-1}$ and $q \vee q^{-1}$ are congruences of the algebra $(A,\text{Pol} q)$ and $q \cap q^{-1} \subseteq q \subseteq q \vee q^{-1}$, we get $q \cap q^{-1} = \Delta$ and $q \vee q^{-1} = \nabla$, thus $q$ is a connected partial order. The “if”-part of (1) follows from 2.5(iii) and 1.3.2.

(2) Let $(A,\text{Pol} q)$ be tolerance simple. Then $(A,\text{Pol} q)$ is also congruence simple, and $q$ is a partial order by (1). Since both, $q \circ q^{-1}$ and $q^{-1} \circ q$, are tolerances of $(A,\text{Pol} q)$ and contain $q$, we get $q \circ q^{-1} = q^{-1} \circ q = \nabla$. These equalities imply that any two elements in the poset $(A,q)$ have a common lower bound and an upper bound, i.e. $q$ is a locally bounded partial order (cf. 1.2). The “if”-part of (2) is a known fact (e.g. for proving that $\text{Pol} q$ is a maximal locally closed clone; see [Pösk79, 4.3.7] in case of finite $A$), but also immediately follows from 2.3(B) since $\varepsilon_2 = \varepsilon_{-2} = \nabla$ for locally bounded partial orders.

(3) If $q$ is a linear order then $(A,\text{End} q)$ is tolerance simple because of 2.5(iv). Conversely, let $(A, \text{End} q)$ be tolerance simple. Since $q \cap q^{-1}$ and $q \cup q^{-1}$ are tolerances of the algebra $(A, \text{End} q)$ and $q \cap q^{-1} \subseteq q \subseteq q \cup q^{-1}$, we obtain $q \cap q^{-1} = \Delta$ (thus $q$ is a partial order) and $q \cup q^{-1} = \nabla$, what implies that $q$ is a linear order. \hfill \Box

Recall that an element $a$ of a poset $(P,\leq)$ with greatest element $1 \in P$ is called dual atom (or coatom) if it is covered by 1, i.e. $a < 1$ and $a \leq b < 1$ implies $a = b$.

Since $\text{End} q \subseteq \text{End} q_0$ and $\text{End} q \subseteq \text{End} (q \vee q^{-1})$, by 2.5(i) we get that $\text{End} q$ $(q \in \text{Quord}(A))$ is a dual atom in the poset $\{(\text{End} \sigma \mid \sigma \in \text{Quord}(A)), \subseteq\}$ if and only if either $q \in \{q_0, q \vee q^{-1}\}$ what gives $q_0 = q = q \vee q^{-1}$, or $q_0 = \Delta$.
and \( q \lor q^{-1} = \triangledown \) (cf. Fig. 2.5.1), i.e. either \( q \) is an equivalence or a connected partial order (cf. 1.2).

The next proposition will answer the more general question under which conditions the monoid \( \text{End} q \) is a dual atom in the larger poset \( (\{ \text{End} Q \mid Q \subseteq \text{Rel}^{(2)}(A) \}, \subseteq) \). The greatest element of this poset is \( \text{End} \Delta = \text{End} \triangledown = A^4 \).

**Proposition 2.7.** Let \( q \in \text{Quord}(A) \). Then the following conditions are equivalent:

1. \( \text{End} q \) is a dual atom in \( (\{ \text{End} Q \mid Q \subseteq \text{Rel}^{(2)}(A) \}, \subseteq) \).
2. \( \text{End} q \) is a dual atom in \( (\{ \text{End} q \mid q \in \text{Rel}^{(2)}(A) \}, \subseteq) \).
3. Either \( q \) is a nontrivial equivalence relation, or the algebra \( (A, \text{End} q) \) is tolerance simple and \( q \notin \{ \Delta, \triangledown \} \).
4. Either \( q \) is a nontrivial equivalence relation or \( q \) is a linear order.

**Proof.** (i) and (i)' are trivially equivalent, since \( \text{End} Q \subseteq \text{End} \sigma \) for all \( \sigma \in Q \).

(i)' \( \implies \) (ii): Let \( \text{End} q \) be a dual atom (thus \( q \notin \{ \Delta, \triangledown \} \)), and let \( \sigma \in \text{Tol}(A, \text{End} q) \setminus \{ \Delta, \triangledown \} \), i.e. \( (A, \text{End} q) \) is not tolerance simple. Then \( \text{End} q \subseteq \text{End} \sigma \neq A^4 \), hence \( \text{End} q = \text{End} \sigma \). With 2.3(A) we conclude that either \( \sigma \in \{ q_0, q, q^{-1} \} \) or \( q \cup q^{-1} \subseteq \sigma \).

Assume by contradiction that \( q \) is not an equivalence, i.e. \( q \) is not symmetric. As now the cases \( \sigma = q \) and \( \sigma = q^{-1} \) are excluded, let \( \sigma = q_0 \). Then \( \text{End} q = \text{End} q_0 \) and therefore \( q \in \text{Inv}^{(2)}(A, \text{End} q_0) \). Since \( q_0 \) is an equivalence, by 2.5(v) we get \( q \in \{ \Delta, \triangledown, q_0 \} \) — a contradiction. Thus \( \sigma = q_0 \) is also excluded and therefore we must have \( q \cup q^{-1} \subseteq \sigma \). Finally, as \( q \) is not symmetric, there exists \((a, b) \in q\) such that \((b, a) \notin q\). Then \((b, a) \in q^{-1} \subseteq \sigma \) and \((a, b) \in \sigma \). According to 1.4, the mapping

\[
    f(x) := \begin{cases} 
        b & \text{if } x = a \\
        a & \text{otherwise}
    \end{cases}
\]

belongs to \( \text{End} \sigma = \text{End} q \). But \((f(a), f(b)) = (b, a) \notin q\) contradicts \((a, b) \in q\) and \( f \triangleright q \). This shows that \( q \) must be an equivalence relation whenever \((A, \text{End} q)\) is not tolerance simple.

(ii) \( \implies \) (iii): Let \((A, \text{End} q)\) be tolerance simple and \( q \notin \{ \Delta, \triangledown \} \). Then \( q \) is a linear order by 2.6(3).

(iii) \( \implies \) (i): Suppose that \( \text{End} q \subseteq \text{End} \sigma \neq A^4 \) for some \( \sigma \subseteq A \times A \). Then \( \sigma \in \text{Inv}^{(2)}(A, \text{End} q) \) and \( \sigma \notin \{ \Delta, \triangledown \} \), hence Corollary 2.5 (iv) and (v) imply that \( \sigma \in \{ q, q^{-1} \} \). Thus \( \text{End} \sigma = \text{End} q = \text{End} q^{-1} \), and this proves that \( \text{End} q \) is a dual atom. \( \square \)

### 2.8. Problems

The maximal clones (dual atoms in the clone lattice) on a finite set are known (see e.g. [Ros70]), among them are the clones \( \text{Pol} \varrho \) where \( \varrho \) is a bounded partial order or an equivalence relation on \( A \). Since \( \text{Pol} q \subseteq \text{Pol} q_0 \cap \text{Pol}(q \lor q^{-1}) \) (this follows from e.g. 2.3), it is clear that in general \( \text{Pol} q \) is not maximal for a quasiorder \( q \). There are examples \( q \) where
Pol $q$ is a so-called submaximal clone (e.g. $q = \Delta \cup \{(0,1)\}$ for $A = \{0,1,2\}$; see [Lau82] or [Lau06, Thm. 14.1.8]). We conclude with the problem to find all clones in the interval $[\text{Pol} q, \text{Op}(A)]$ (in the lattice of all clones) as well as to find all monoids in the interval $[\text{End} q, \text{Op}^{(1)}(A)]$ (in the lattice of all submonoids of $\text{Op}^{(1)}(A)$).

**Remarks 2.9.** We mention here the connection between quasiorders and closure operators and corresponding Alexandroff topologies (i.e. topologies where arbitrary intersections of open sets are open). Namely (see e.g. [Ern05] or [Ern91]), for a quasiorder $q \in \text{Quord}(A)$, the operator $\varphi : \mathcal{P}(A) \to \mathcal{P}(A) : B \mapsto \varphi(B)$ is a closure operator (the so-called down-closure) where $\varphi(B) := \{a \in A \mid \exists b \in B : (a, b) \in q\}$ is the “down-set” of $B$. Vice versa, for a closure operator $\varphi$ we get a quasiorder via $q := \{(a, b) \mid a \in \varphi(\{b\})\}$. This establishes a correspondence between quasiorders and closure operators (which is injective on quasiorders). The corresponding (upper) Alexandroff topology $T(A, q)$ is given by the closed sets of $\varphi$, i.e. the closed sets of $T$ are the down-sets $\varphi(B)$ of $q$ (and the open sets are the “up-sets”). The so-called specialization order $q(T)$ of a topology $T$, which in general is a quasiorder, is given by $(x, y) \in q(T) : \iff \mathcal{O}_x \subseteq \mathcal{O}_y$ (where $\mathcal{O}_x$ denotes the set of all open sets containing $x$). Then $q(T(A, q)) = q$ for any quasiorder $q$; moreover, the endomorphisms of $q$ are exactly the mappings $f : A \to A$ which are continuous with respect to $T(A, q)$.

Therefore, all our results can be easily transformed to corresponding results about Alexandroff topologies $T$ on $A$ and continuous mappings $\text{End}(A, T)$.

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