AN APPROACH TO ORTHOMODULAR LATTICES VIA LATTICES WITH AN ANTITONE INVOLUTION

IVAN CHAJDA* — SÁNDOR RADELECKZI**

(Communicated by Anatolij Dvurečenskij)

ABSTRACT. Several characterizations of orthomodular lattices based on properties of an antitone
involution or on sectional antitone involutions are given. Another approach is based on properties of
the implication operation which can be derived in every orthomodular lattice but can be also added to
basic operations of a bounded lattice. Finally, we introduce the so-called weakly orthomodular lattice
as a generalization of orthomodular lattices.

1. Introduction

Algebraic axiomatization of the logic of quantum mechanics is provided via orthomodular lat-
tices, see e.g. [13], [3] or [1]. The algebraic theory of orthomodular lattices is captured in mono-
graphs [2] and [14]. In this paper we present several characterizations of orthomodular lattices. We
characterize them among lattices with an antitone involution or among lattices having sectionally
antitone involutions in every upper interval. Let us note that these lattices were treated already
in [5], see also [9] and [11] for an overview and applications. Another approach presented here
is by means of an additional binary operation which is the implication. It is worth noticing that
an implication in orthomodular lattices was investigated also in [1], [7], [8], [12] and [15]. More-
over, for orthocomplemented lattices it was treated in [7] — see also [6] as a source of results on
orthocomplemented lattices.

Finally, we will get a characterization of weakly orthomodular lattices which constitute a gen-
eralization of orthomodular lattices. These lattices play an important role in logic of quantum
mechanics because the lattice of closed linear subspaces of a Hilbert space is orthomodular in gen-
eral and it is a modular orthocomplemented lattice if the space is of finite dimension, see e.g. [3],

By a lattice with an antitone involution we mean an algebra $L = (L; \lor, \land, \prime, 0, 1)$ of type
$(2, 2, 1, 0, 0)$ such that $(L; \lor, \land, 0, 1)$ is a bounded lattice with the least element $0$ and the greatest
element $1$ and $\prime$ is a unary operation on $L$ satisfying the following two conditions:

$$x'' = x \quad \text{(i.e. the mapping } x \mapsto x' \text{ is an involution),} \tag{1}$$

$$x \leq y \Rightarrow y' \leq x' \quad \text{(} x \mapsto x' \text{ is antitone).} \tag{2}$$

By an orthocomplemented lattice (see [2] or [11]) is meant a lattice $L$ with an antitone involution
which is a complementation, i.e. it satisfies

$$x \land x' = 0, \tag{3}$$

2010 Mathematics Subject Classification: Primary 06B05, 06C15; Secondary 03G25.

Keywords: bounded lattice, antitone involution, sectional antitone involutions, orthomodular lattice, orthocom-
plemented lattice, weakly orthomodular lattice, implication.

The work was supported by the Austrian Science Fund (FWF), project I 1923-N25, and Czech Science Foundation
(GAČR), project 15-34697L.
which is equivalent to \( x \lor x' = 1 \), due to De Morgan laws:

\[
(x \lor y)' = x' \land y' \quad \text{and} \quad (x \land y)' = x' \lor y'.
\]

An orthocomplemented lattice is called orthomodular (see [2] or [11]) if it satisfies the condition

\[
x \leq y \implies x \lor (x' \land y) = y,
\]

which is equivalent to

\[
x \leq y \implies y \land (y' \lor x) = x
\]

and it is also equivalent to the orthomodular identity:

\[
((x \lor y) \land x') \lor x = x \lor y.
\]

Let \((L; \lor, \land, 0, 1)\) be a bounded lattice. For each \(a \in L\) the interval \([a, 1] = \{x \in L \mid a \leq x \leq 1\}\) is called a section. We say that the algebra \(\mathcal{L} = (L; \lor, \land, \{a\} a \in L\; 0, 1)\) is a lattice with sectionally antitone involutions (c.f. [5]) if for each \(a \in L\) there exists an antitone involution \(x \mapsto x^a\) of the sublattice \((a,1]; \lor, \land\).

2. Orthomodular lattices as particular lattices with antitone involution

In the introduction orthomodular lattices were defined as orthocomplemented lattices satisfying (OML) or equivalently (OMI). It is worth noting that in the case of an arbitrary lattice with an antitone involution the law \(x \lor x' = 1\) follows from each of these laws, since both (OML) and (OMI) imply \(1 = x \lor (1 \land x')\). In addition, we prove the following characterization:

**Theorem 2.1.** Let \(\mathcal{L} = (L; \lor, \land, 0, 1)\) be a lattice with an antitone involution. Then the following assertions are equivalent:

(a) \((x \lor y)' \lor x = (x \lor y) \land x' \lor y\), for all \(x,y \in L\).

(b) \((x \lor y)' \lor x \geq x \lor y\), for all \(x,y \in L\).

(c) \(\mathcal{L}\) is an orthomodular lattice.

**Proof.** Clearly, \((x \lor y)' \lor x = (x \lor y) \land x' \lor y\). If \(\mathcal{L}\) is orthomodular then \((x \lor y) \land x' \lor x = x \lor y\), directly by (OMI). Hence (c) \(\implies\) (b).

(b) \(\implies\) (a). Since \((x \lor y) \land x' \lor x \leq x \lor y\), we have \((x \lor y)' \lor x' \lor x \leq x \lor y\). Thus (b) yields \((x \lor y)' \lor x' \lor x = x \lor y\), and this clearly implies (a).

(a) \(\implies\) (c). Assume that \(\mathcal{L}\) satisfies (a), and set

\[
z = ((x \lor y) \land x') \lor x = ((y \lor x) \land y') \lor y.
\]

Then clearly, \(x, y \leq z\) and hence \(x \lor y \leq z\). On the other hand \((x \lor y) \land x' \lor x \leq x \lor y\). Altogether we obtain \((x \lor y) \land x' \lor x = z = x \lor y\). Thus (OMI) is satisfied by \(\mathcal{L}\), and hence \(\mathcal{L}\) is orthomodular.

Let \(\mathcal{L} = (L; \lor, \land, 0, 1)\) be a lattice with an antitone involution. For each section \([a, 1], a \in L\), we define the mapping \(x \mapsto x^a := x' \lor a, x \in [a, 1]\). Of course, this mapping is antitone because \(x \leq y\) for \(x,y \in [a,1]\) implies

\[
y^a = y' \lor a \leq x' \lor a = x^a.
\]

We prove the following:

**Theorem 2.2.** Let \(\mathcal{L}\) be a lattice with an antitone involution. Then the following assertions are equivalent:

(a) For each \(a,x \in L\) with \(a \leq x\), \(x' \lor a\) is a complement of \(x\) in the section \([a,1]\).

(b) For each \(a \in L\), the mapping \(x \mapsto x^a\), \(x \in [a,1]\) is extensive, i.e. \(x^a \geq x\), for all \(x \in [a,1]\).

(c) \(\mathcal{L}\) is an orthomodular lattice.
Proof. Assume that \( L \) is orthomodular and let \( a, x \in L \), \( a \leq x \). Clearly, \((x' \lor a) \lor x = 1\) and \((x' \lor a) \land x = a\) by the orthomodular law. Hence \( x' \lor a \) is a complement of \( x \) in \([a, 1]\), i.e. \((c) \implies (a)\).

Further, \( x^{aa} = (x^a)^{c} = (x' \lor a)^{c} \lor a = a \lor (a' \land x) = x \), for each \( x \in [a, 1] \) by (OML). Hence \((c) \implies (b)\).

Conversely, assume that \((a)\) holds. Then for any \( a, x \in L \) with \( a \leq x \) we have \((x' \lor a) \land x = a\), which is the orthomodular law, i.e. \((a) \implies (c)\).

Assume now \((b)\). Then for \( x' \lor y \in [x, 1] \) we obtain:

\[ x \lor y \leq (x \lor y)_{x'} = ((x \lor y)' \lor x)_{x'} \lor x. \]

Since the condition \((b)\) of Theorem 2.1 is satisfied, we obtain that \( L \) is an orthomodular lattice, that is, \((b) \implies (c)\).

The following consequence is immediate:

**Corollary 2.3.** Let \( L = (L; \lor, \land, ^{t}, 0, 1) \) be a lattice with an antitone involution. Then \( x \mapsto x' \lor a \), \( a \in [a, 1] \) is an antitone involution for each section \([a, 1] \subseteq L, a \in L\), if and only if \( L \) is orthomodular.

### 3. Lattices with sectionally antitone involutions

Let \( L = (L; \lor, \land, \{ a \mid a \in L \}, 0, 1) \) be a lattice with sectionally antitone involutions and let \( \cdot \) denote the unary operation \( x \mapsto x^{0}, x \in L \) corresponding to \( 0 \in L \). If for any \( a, b, x \in L \) with \( a \leq b \leq x \) the identity

\[ x^{b} = x^{a} \lor b \]  

(CC)
holds, then we say that \( L \) satisfies the compatibility condition (see \[1\] and \[8\]).

**Proposition 3.1.** Let \( L = (L; \lor, \land, \{ a \mid a \in L \}, 0, 1) \) be a lattice with sectionally antitone involutions. Then the following are equivalent:

1. \((L; \lor, \land, ^{t}, 0, 1)\) is an orthomodular lattice and \( x^{a} = x' \lor a \), for all \( a, x \in L \) with \( a \leq x \).
2. \( L \) satisfies the compatibility condition.

**Proof.**

(i) \( \implies \) (ii). Let \( a \leq b, a, b \in L \). If (i) holds then \( x^{b} = x' \lor b = x' \lor a \lor b = x^{a} \lor b \), for all \( x \geq b \), i.e. (CC) is satisfied.

(ii) \( \implies \) (i). (CC) implies \( x^{a} = x^{0} \lor a = x' \lor a \). Hence \( x \mapsto x' \lor a \), \( a \in [a, 1] \), is an antitone involution in each section \([a, 1] \), \( a \in L \). Thus \((L; \lor, \land, ^{t}, 0, 1)\) is an orthomodular lattice by Corollary 2.3.

Hence orthomodular lattices can be viewed as retracts \((L; \lor, \land, ^{0}, 0, 1)\) of the lattices with sectionally antitone involutions satisfying the compatibility condition.

If \( L \) is a lattice with sectionally antitone involutions, then in view of \[9\] we can define a binary operation \( \rightarrow \) on \( L \) as follows:

\[ x \rightarrow y := (x \lor y)^{y}. \]

Then clearly \((x \lor y) \rightarrow y = x \rightarrow y \) and \( x^{a} = x \rightarrow a \) for all \( a, x \in L \) with \( x \geq a \). Let us define \( x' := x^{0} \). This means that \( x' = x \rightarrow 0 \), for all \( x \in L \). Now, observe that by using operation \( \rightarrow \), the compatibility condition can be rewritten as follows:

\[ x \rightarrow b = (x \rightarrow a) \lor b \]  

for all \( a, b, x \in L \) with \( a \leq b \leq x \).

This implies \( x \rightarrow y = (x \lor y) \rightarrow y = ((x \lor y) \rightarrow 0) \lor y = (x \lor y)' \lor y = (x' \land y') \lor y. \)
Conversely, suppose that the identity
\[(x \lor y) \Rightarrow y = ((x \lor y) \Rightarrow 0) \lor y\] (A)
holds. Then for any \(a, b, x \in L\) with \(a \leq b \leq x\) we obtain:
\[x^a \lor b = (x \Rightarrow a) \lor b = ((x \lor a) \Rightarrow a) \lor b = ((x \lor a) \Rightarrow 0) \lor a \lor b\]
\[= (x \Rightarrow 0) \lor b = ((x \lor b) \Rightarrow 0) \lor b = (x \lor b) \Rightarrow b = x^b.\]
This means that the condition (CC) holds for \(L\). Hence by applying Proposition 3.1 we obtain:

**Corollary 3.2.** Let \(L\) be a lattice with sectionally antitone involutions, and define \(x' := x^0\) and \(x \Rightarrow y := (x \lor y)^\prime\), for all \(x, y \in L\). Then \(L\) satisfies identity (A) if and only if \((L; \lor, \land, ^\prime, 0, 1)\) is an orthomodular lattice and \(x \Rightarrow y = (x' \land y') \lor y\).

## 4. Lattices with implication

First, we note that it is implicit in \([3]\) (see also \([11]\)) that in an orthomodular lattice \(L\) there can exist several binary operations satisfying \(x \Rightarrow y = x^0\) for all \(x, y \in L, x \geq y\), and moreover, each of them satisfies the identities:
\[((x \lor y) \Rightarrow x) \Rightarrow x = x \lor y,\]
\[((x \lor y) \Rightarrow 0) \lor (y \Rightarrow 0) = y \Rightarrow 0.\]

**Theorem 4.1.** Let \((L; \lor, \land, 0, 1)\) be a bounded lattice and \(\Rightarrow\) a binary operation on \(L\) such that
\[(x \lor y) \Rightarrow y = ((x \lor y) \Rightarrow 0) \lor y.\]
Define \(x' = x \Rightarrow 0\), for all \(x \in L\). Then the following conditions are equivalent:
(i) \(\Rightarrow\) satisfies the identities (B) and (C).
(ii) \(L = (L; \lor, \land, ^\prime, 0, 1)\) is an orthomodular lattice.

**Proof.**
(i) \(\Rightarrow\) (ii). Let \(x \geq y, x, y \in L\). Then \(x = x \lor y\) and \(x' = (x \lor y)^\prime = (x \lor y) \Rightarrow 0 \leq y \Rightarrow 0 = y^\prime\), by (C). Substituting \(x = 0\) into (B) we also get \(y'' = (y \Rightarrow 0) \Rightarrow 0 = y\). Hence \(L\) is a lattice with an antitone involution.

Further, \((x \lor y) \Rightarrow x = ((x \lor y) \Rightarrow 0) \lor x = ((x \lor y)^\prime \lor x) \geq x\). Hence we get \(((x \lor y) \Rightarrow x) \Rightarrow x = ((x \lor y)^\prime \lor x) \lor x = ((x \lor y)^\prime \lor x)^\prime \lor x.\) Now (B) implies \(((x \lor y)^\prime \lor x)^\prime \lor x = x \lor y\), which shows that \(L\) is an orthomodular lattice.

Conversely, suppose that \(L\) is an orthomodular lattice and
\[(x \lor y) \Rightarrow y = (x \lor y)^\prime \lor y\ and \ x' = x \Rightarrow 0 \ for \ all \ x, y \in L.\]
Then \((x \lor y)^\prime \leq y^\prime\), hence \(((x \lor y) \Rightarrow 0) \lor (y \Rightarrow 0) = (x \lor y)^\prime \lor y^\prime = y = y \Rightarrow 0, i.e. (C) holds.\ We also get
\[((x \lor y) \Rightarrow x) \Rightarrow x = ((x \lor y)^\prime \lor x) \lor x = ((x \lor y) \land x') \lor x = x \lor y\ by \ (OMI), which is just condition (B).\]

**Corollary 4.2.** Let \(L = (L; \lor, \land, ^\prime, 0, 1)\) be a lattice with an antitone involution and \(\Rightarrow\) a binary operation on \(L\) such that for any \(x, y \in L\) with \(x \geq y\), \(x \Rightarrow y = x' \Rightarrow y\). Then \(L\) is an orthomodular lattice if and only if \(\Rightarrow\) satisfies (B).

**Proof.** By our assumption we have \(x' = x \Rightarrow 0\) and \((x \lor y) \Rightarrow y = (x \lor y)^\prime \lor y = ((x \lor y) \Rightarrow 0) \lor y.\) If \(L\) is an orthomodular lattice, applying Theorem 4.1, we obtain that \(\Rightarrow\) satisfies condition (B).

Conversely, suppose that (B) is satisfied by \(\Rightarrow\). Because of 
\[(x \lor y)^\prime \leq y^\prime\ we get that \(((x \lor y) \Rightarrow 0) \lor (y \Rightarrow 0) = (x \lor y)^\prime \lor y^\prime = y' = y \Rightarrow 0,\ and \ hence (C) is also satisfied by \(\Rightarrow\)\]. Now, by applying Theorem 4.1 once more we obtain that \(L\) is an orthomodular lattice.
AN APPROACH TO ORTHOMODULAR LATTICES VIA LATTICES WITH AN ANTITONE INVOLUTION

Assume now that $L$ is an orthomodular lattice. It is well-known (see e.g. [2] or [12]) that there are just six possibilities to define an implication $x \rightarrow y$ as a term of the algebra $L = (L; \lor, \land', \cdot)$ such that these coincide with the Boolean implication in any Boolean subalgebra of $L$. These possibilities are as follows:

1. $x \rightarrow y = x' \lor y$ (the usual “Boolean” implication),
2. $x \rightarrow y = (x' \lor y) \land (x \lor (x' \land y') \lor (x' \land y))$,
3. $x \rightarrow y = (x' \land y') \lor (x' \land y) \lor (x \land y)$,
4. $x \rightarrow y = x' \lor (x \land y)$,
5. $x \rightarrow y = (x' \lor y) \land (y' \lor (y \land x') \lor (y \land x))$,
6. $x \rightarrow y = (x' \land y') \lor y$.

It is known (see e.g. [2]) that all these implications satisfy the identity

\[ x \rightarrow (x \lor y) = 1. \]

**Remark 1.** These operations “→” can be defined in an arbitrary lattice with an antitone involution, and it is easy to see that they all coincide for $x \geq y$. For instance, in case of (4) and (6) it is immediate that $x \rightarrow y = x' \lor y$ whenever $x \geq y$. Now, let us check the implication as given in (5):

If $x \geq y$ then $x \rightarrow y = (x' \lor y) \land (y' \lor (y \land x')) \lor (y \land x) = (x' \lor y) \land (y' \lor (y \land x'))$. Since $x' \leq y'$, we get $x' \lor y \leq y' \lor y$, whence $(x' \lor y) \land (y' \lor (y \land x')) = x' \lor y$.

In view of Remark 1, by using Corollary 4.2 we obtain the following:

**Corollary 4.3.** Let $L = (L; \lor, \land', 0, 1)$ be a lattice with an antitone involution and let $\rightarrow$ be one of the implications defined by formulas (1)–(6). Then $L$ is an orthomodular lattice if and only the identity

\[ ((x \lor y) \rightarrow x) \rightarrow x = x \lor y. \]

is satisfied.

The following result shows that in the case when $\rightarrow$ is a binary term of the algebra $(L; \lor, \land')$, then it is determined completely by the conditions (A), (B), (C) and (D).

**Theorem 4.4.** Let $(L; \lor, \land, 0, 1)$ be a bounded lattice, $\rightarrow$ a binary operation on $L$, and define $x' = x \rightarrow 0$, for all $x \in L$. Then the following assertions are equivalent:

1. $\rightarrow$ is a term of the algebra $(L; \lor, \land')$ and it satisfies the conditions (A), (B), (C) and (D),
2. $L = (L; \lor, \land', 0, 1)$ is an orthomodular lattice and $\rightarrow$ is one of the implications defined by formulas (1)–(6).

**Proof.**

(ii) $\Rightarrow$ (i). The operation $\rightarrow$ satisfies (D) and, in view of Remark 1, $x \rightarrow y = x' \lor y$, for all $x, y \in L$. Hence we get $(x \lor y) \rightarrow y = (x \lor y)' \lor y = ((x \lor y) \rightarrow 0) \lor y$, which is (A). Since $L$ is an orthomodular lattice, in view of Theorem 4.1, $\rightarrow$ satisfies also (B) and (C).

(i) $\Rightarrow$ (ii). Suppose $\rightarrow$ satisfies (A), (B), (C) and (D). In view of Theorem 4.1, $L$ is an orthomodular lattice. Now let $B = (B; \lor, \land', 0, 1)$ be a Boolean subalgebra of $L$. Since $\rightarrow$ is a term of $L$, its restriction to $B$ is also a term of $B$. By the definition of $'$ we have $1 \rightarrow 0 = 1' = 0$ and $0 \rightarrow 0 = 0' = 1$. The identity (D) yields also $0 \rightarrow 1 = 1 \rightarrow 1 = 1$. Hence the restriction of $x \rightarrow y$ to $\{0, 1\}$ coincides with the restriction of $x' \lor y$ to $\{0, 1\}$. Because $B$ is a Boolean algebra and $x \rightarrow y$ and $x' \lor y$ are its terms, this means that $x \rightarrow y = x' \lor y$, for all $x, y \in B$. Finally, observe that those terms of $L$ that coincide with the Boolean implication $x' \lor y$ on any Boolean subalgebra of $L$ are only the implications given by formulas (1)–(6). Indeed, the set of all terms in two variables of an orthomodular lattice forms the free orthomodular lattice with two free generators $F_{OML}(x, y)$. As it is shown in [2; Theorem 2.8], $F_{OML}(x, y)$ is the direct product of the orthomodular lattice $MO_2$ (having 6 elements) and the free Boolean algebra with two free generators $2^4$ (having 16 elements).
Hence, for every Boolean term \( t(x, y) \) there exist six different binary terms in \( \mathcal{F}_{\text{OML}}(x, y) \) which coincide with \( t(x, y) \) in the free Boolean algebra with two generators. Thus, there are six binary terms of \( \mathcal{L} \) which coincide with the Boolean implication \( x' \lor y \) on any Boolean subalgebra of \( \mathcal{L} \) (see also [12]). It is an easy exercise to check that they are just those given by formulas (1)–(6).

\[ \square \]

5. Weakly orthomodular lattices

An axiomatization of the logic of quantum mechanics is usually provided by means of selfdual operators of a suitable Hilbert space \( H \) where the physical events are measured (see e.g., [15]). It is well-known that every such operator is determined by a certain closed linear subspace of \( H \) and hence it is asked to describe the structure of all closed linear subspaces of \( H \). This was firstly treated by K. Husimi [13] and also by G. Birkhoff and J. von Neumann in [4]. It was shown in [13] that the set of all closed linear subspaces of \( H \) is an orthomodular lattice in general, and a modular orthocomplemented lattice if \( H \) is of finite dimension. Hence in the last case we obtain a particular modular lattice with an antitone involution.

In [10] the authors have presented a common generalization of orthocomplemented lattices and MV-algebras. An other interesting approach could be to find a common generalization of orthomodular lattices and modular lattices with an antitone involution:

A lattice \( \mathcal{L} \) with an antitone involution is called weakly orthomodular if it satisfies the axiom

\[ x \leq y \implies x \lor (y \land x') = y \land (x \lor x'). \]  

(W)

**Remark 2.** Every orthomodular lattice is weakly orthomodular because \( x \lor x' = 1 \), and hence (W) is equivalent to (OML). Every modular lattice with an antitone involution is also weakly orthomodular because (W) follows by the modular identity.

**Example.** If \( \mathcal{A} \) is an orthomodular lattice which is not modular and \( \mathcal{B} \) is a (nontrivial) distributive lattice with an antitone involution which is not a complementation, then the direct product \( \mathcal{A} \times \mathcal{B} \) is a weakly orthomodular lattice which is neither orthomodular nor modular, by its construction. In particular, consider \( \mathcal{A} \) to be the orthomodular lattice which is the horizontal sum of 8-element Boolean algebra with 4-element Boolean algebra and \( \mathcal{B} \) to be the three element chain (in which the antitone involution is determined in a unique way). Since \( \mathcal{A} \) is not modular and \( \mathcal{B} \) is not a complemented lattice the product \( \mathcal{A} \times \mathcal{B} \) is neither modular nor orthomodular, but it is still a weakly orthomodular lattice.

**Theorem 5.1.** A lattice \( \mathcal{L} \) with an antitone involution is weakly orthomodular if and only if \( \mathcal{L} \) does not contain a sublattice depicted in Figure 1, where \( u = x \lor (y \land x') \) and \( v = y \land (x \lor x') \).

**Figure 1.**

\[ \begin{array}{c}
\begin{array}{c}
\text{Figure 1.}
\end{array}
\end{array} \]

**Proof.** Of course, \( x \leq y \) implies \( x \lor (y \land x') \leq y \land (x \lor x') \). Hence \( x \leq u \leq v \leq y \), for all \( x, y \in \mathcal{L} \) with \( x \leq y \). If \( \mathcal{L} \) contains a sublattice given in Figure 1, then \( x \leq x \lor (y \land x') = u < v = y \land (x \lor x') \leq y \) which contradicts (W).
Conversely, suppose that \( L \) is not weakly orthomodular. Then there exist \( x, y \in L \) with \( x < y \) such that \( u < v \). Then
\[
u \lor x' = x \lor (y \land x') \lor x' = x \lor x' \quad \text{and} \quad v \lor x' \geq u \lor x' = x \lor x',
\]
but \( v = y \land (x \lor x') \leq x \lor x' \). Thus we get \( v \lor x' = x \lor x' = u \lor x' \).

Analogously,
\[
v \land x' = y \land (x \lor x') \land x' = y \land x' \quad \text{and} \quad u \land x' \leq v \land x' = y \land x'.
\]
However, \( u = x \lor (y \land x') \geq y \land x' \), whence we get \( u \land x' \geq (y \land x') \land x' = y \land x' \), i.e. \( u \land x' = y \land x' = v \lor x' \). Hence \((\{y \land x', u, v, x', x \lor x'\}, \lor, \land)\) is a sublattice of \( L \). Let us show that all these elements are different:

Indeed, since \( x \lor (y \land x') = u \), the equality \( u = y \land x' \) would imply \( x \leq u \leq x' \), whence we would get \( v = y \land (x \lor x') = y \land x' = u \), contrary to our assumption. Thus we have \( y \land x' < u \).

By definition \( v = y \land (x \lor x') \leq y \). If \( v = x \lor x' \), then we get \( x' \leq x \lor x' \leq y \), and this implies \( u = x \lor (y \land x') = x \lor x' = v \), contrary to our assumption. This means that \( v < x \lor x' \).

Hence \( y \land x' < u < v < x \lor x' \). Similarly, \( x' \in \{y \land x', u, v, x \lor x'\} \) would lead to a contradiction. Thus, we have shown that \((\{y \land x', u, v, x', x \lor x'\}, \lor, \land)\) is a sublattice isomorphic to the sublattice depicted in Figure 1.

**Proposition 5.2.** Let \( L \) be a weakly orthomodular lattice. Then the following assertions are equivalent:

(i) The mapping \( x \mapsto x^a := x' \lor a, \ x \in [a, 1] \), is surjective for all \( a \in L \).

(ii) \( x \mapsto x^a, \ x \in [a, 1] \), is an antitone involution for each \( a \in L \).

(iii) \( L \) is an orthomodular lattice.

**Proof.**

(ii) \( \implies \) (i) is clear, and (iii) \( \implies \) (ii) holds by Corollary 2.3.

Prove (i) \( \implies \) (iii). Since the mapping \( x \mapsto x^a \) is antitone, we have \( a^n \geq x^a \) for all \( x \geq a, \ a, x \in L \). Because the mapping \( x \mapsto x^a, \ x \in [a, 1] \) is surjective, \( b^a = 1 \) for some \( b \in [a, 1] \). Then \( 1 \geq a^n \geq b^a = 1 \) implies \( a \lor a' = a^n = 1 \), for all \( a \in L \). Hence, in view of condition (W), \( L \) is an orthomodular lattice.

Now, let \( L = (L; \lor, \land', 0, 1) \) be a weakly orthomodular lattice, and consider the set
\[
X = \{x \lor x' \mid x \in L\},
\]
and the congruence \( \alpha = \theta(X) \) of \( L \), generated by \( X^2 \).

**Proposition 5.3.** \( \alpha \) is the least congruence of \( L \) having the property that \( L/\alpha \) is an orthomodular lattice.

**Proof.** Clearly, \( [x]_\alpha \mapsto [x']_\alpha \) is a well-defined involution on the factor lattice \((L/\alpha, \lor, \land)\). Set \( [x]_\alpha := [x']_\alpha \), for all \( x \in L \). Since \( 1 = 1 \lor 1' \in X \), we have \( (x \lor x', 1) \in \alpha \), for all \( x \in L \). Then
\[
[x]_\alpha \lor [x']_\alpha = [1]_\alpha.
\]
As \( (x \lor x') = x'' \land x' = x \land x' \) and \( 1' = 0 \) imply \( (x \land x', 0) \in \alpha \), we also get \( [x]_\alpha \land [x']_\alpha = [0]_\alpha \). Thus \( [x]_\alpha \mapsto [x']_\alpha \) is an orthocomplementation on \((L/\alpha, \lor, \land)\).

Assume that \( [x]_\alpha \leq [y]_\alpha \). Then \( [x]_\alpha = [x]_\alpha \land [y]_\alpha = [x \land y]_\alpha \). Let \( t = x \land y \). Then \( [x]_\alpha = [t]_\alpha \) implies \( [x]_\alpha = [t']_\alpha = [t']_\alpha \). Clearly, \( [t \lor t']_\alpha = [1]_\alpha \). Since \( t \leq y \), by using axiom (W) we obtain:
\[
[y]_\alpha \land [t \lor t']_\alpha = [t \lor t']_\alpha = [t \lor (y \land t')]_\alpha = [t \lor (y \lor t')]_\alpha = [y \land (t \lor t')]_\alpha = [y]_\alpha.
\]
This means that the factor algebra \((L/\alpha, \lor, \land', [0]_\alpha, [1]_\alpha)\) is an orthomodular lattice. Now, let \( \beta \subseteq L^2 \) be a congruence of \( L \) with the property that \( L' = (L/\beta, \lor, \land', 0_B, 1_B) \) is an orthomodular lattice. Then the natural projection \( \pi: L \to L/\beta \), \( x \mapsto [x]_\beta \) is a homomorphism of the algebra \( L \) onto \( L' \) and \( \beta = ker \pi \). We will show that \( \alpha \subseteq ker \pi \).

779
Indeed, we have \( \pi(1) = 1_\beta \) by definition, and \( \pi(x \lor x') = \pi(x) \lor \pi(x)' = 1_\beta \), because \( x \mapsto x' \) is an orthocomplementation operation in \( L' \). Hence \( (x \lor x', 1) \in \ker \pi \), for all \( x \in L \). Thus \( (x \lor x', y \lor y') \in \ker \pi \), for all \( x, y \in L \). However, this result means that \( X^2 \subseteq \ker \pi \), and we obtain \( \alpha = \theta(X) \subseteq \ker \pi = \beta \). □

The following consequence is obvious:

**Corollary 5.4.** Any modular lattice \( L \) with an antitone involution has a least congruence \( \alpha \) such that \( L/\alpha \) is a modular ortholattice.

**Acknowledgement.** The authors would like to thank the referees for their valuable and constructive comments.

**REFERENCES**


Received 17. 9. 2013
Accepted 9. 2. 2014

*Department of Algebra and Geometry
Faculty of Science
Palacký University
17. Listopadu 12
77146 Olomouc
CZECH REPUBLIC
E-mail: ivan.chajda@upol.cz

**Institute of Mathematics
University of Miskolc
H-3515 Miskolc-Egyetemváros
HUNGARY
E-mail: matradi@uni-miskolc.hu