Representation of integral quantales by tolerances

KALLE KAARLI AND Sándor Radeleczki

Dedicated to the memory of E. Tamás Schmidt

Abstract. The central result of the paper claims that every integral quantale $Q$ has a natural embedding into the quantale of complete tolerances on the underlying lattice of $Q$. As an application, we show that the underlying lattice of any integral quantale is distributive in 1 and dually pseudocomplemented. Besides, we exhibit relationships between several earlier results. In particular, we give an alternative approach to Valentini’s ordered sets and show how the ordered sets are related to tolerances.

1. Introduction

Quantales are certain partially ordered algebraic structures that generalize locales (point free topologies) as well as various multiplicative lattices of ideals from ring theory and functional analysis ($C^*$-algebras, von Neumann algebras), see e.g., [17].

Definition 1.1. A quantale is an algebraic structure $Q = (Q, \lor, \cdot)$ such that $(Q, \leq)$ is a complete lattice (induced by the join operation $\lor$) and $(Q, \cdot)$ is a semigroup satisfying

$$a \cdot (\bigvee\{b_i \mid i \in I\}) = \bigvee\{ab_i \mid i \in I\},$$

$$\bigvee\{b_i \mid i \in I\} \cdot a = \bigvee\{b_ia \mid i \in I\}$$

for all $a \in Q$ and $b_i \in Q$, $i \in I$.

Remark 1.2. As usual, we often write $xy$ instead of $x \cdot y$.

In what follows the underlying lattice of a quantale $Q$ is denoted by $L$. Thus, the universes of $Q$ and $L$ coincide: $Q = L$. The smallest and the greatest element of $L$ will be denoted by 0 and 1, respectively. As a consequence of
(1.1) and (1.2), the partial order \( \leq \) of the lattice \( L \) is \textit{compatible with} the multiplication, i.e., for \( x, y, u, v \in L \),

\[
x \leq y \quad \text{and} \quad u \leq v \implies xu \leq yv.
\]  

(1.3)

Two quantales \( Q_1 \) and \( Q_2 \) are called \textit{isomorphic} \( (Q_1 \cong Q_2) \) if there is a bijection \( \phi: Q_1 \to Q_2 \) that is an isomorphism of multiplicative semigroups and preserves all joins. If we replace the multiplication operation of a quantale \( Q \) by its opposite \( \ast \) \( (x \ast y = y \cdot x \text{ for all } x, y \in Q) \) we obtain a new quantale called the \textit{opposite quantale} of \( Q \) and denoted by \( Q^{\text{op}} \). If \( Q_1, Q_2 \) are quantales such that \( Q_1 \cong Q_2^{\text{op}} \) then we call \( Q_1 \) and \( Q_2 \) \textit{anti-isomorphic}. A quantale isomorphism \( \phi: Q \to Q^{\text{op}} \) such that \( \phi = \phi^{-1} \) is called an \textit{involution} of \( Q \). We call a quantale \textit{involutive} if it has an involution. If \( Q \) is an abstract involutive quantale, we denote its involution operation by \( -^1 \).

A quantale \( Q \) is called \textit{commutative} if the operation \( \cdot \) is commutative, and we say that \( Q \) is \textit{unital} whenever \((Q, \cdot)\) is a monoid. A unital quantale in which \( 1 \) is the multiplicative identity element is called \textit{integral}. Hence in any integral quantale \( 1 \cdot x = x = x \cdot 1 \) holds for all \( x \in L \). For any \( x \in L \), we have also \( 0 \cdot x = 0 = x \cdot 0 \) (this follows from distributive laws (1.1) and (1.2) in case \( I = \emptyset \)).

A quantale \( Q_1 \) is called a \textit{subquantale} of \( Q_2 \) if \( Q_1 \subseteq Q_2 \) and \( Q_1 \) is closed under arbitrary joins and the multiplication in \( Q_2 \). Clearly, if \( Q_1 \) is a subquantale of \( Q_2 \) then they have the common \( 0 \).

Let \( Q \) be a unital quantale with multiplicative identity element \( e \). Then the set \( A = e \downarrow = \{ x \in Q \mid x \leq e \} \) is a subuniverse of \( Q \). The quantale \( A \) is clearly integral, it will be called the \textit{integral part} of \( Q \).

In the literature, there have been introduced several quantales related to a given complete lattice. We collect the information about these quantales and relationships between them, and add some new aspects. Here is the list of our main results.

1. We observe that the compatible complete reflexive binary relations on any complete lattice form an integral quantale. Also, the set of all compatible reflexive binary relations on any finite majority algebra is an integral quantale.

2. We show that the quantale of tolerances introduced in [2], implicitly appeared already in [16].

3. We give a more transparent description of \textit{ordered relations} introduced by Valentini [19] and show how the reflexive ordered relations are connected to tolerances.

4. We prove a Cayley type representation theorem for integral quantales and show how it is related to Valentini’s representation theorem for unital quantales.

5. We show that when applied to frames the former result yields an embedding of a relatively pseudocomplemented lattice into its congruence lattice.
We prove that the underlying lattice of any finite integral quantale is dually pseudocomplemented and distributive in 1.

We follow the standard notation concerning binary relations. If \( R \) and \( S \) are binary relations on a set \( A \) then

\[
R \circ S = \{(x, z) \in A^2 \mid \exists y \in A \ (x, y) \in R, (y, z) \in S\}
\]

is their relational product. The converse of \( R \) will be denoted by \( R^\sim (R^\sim = \{(a, b) \mid (b, a) \in R\}) \). The diagonal or equality relation on \( A \) is \( \Delta = \Delta_A = \{(a, a) \mid a \in A\} \). The total relation on \( A \) is \( \nabla = \nabla_A = A \times A \). It is well known that all binary relations on \( A \) form a monoid with \( \circ \) in role of multiplication and \( \Delta_A \) in role of identity element.

2. Quantales from compatible relations under intersection

Throughout this section, \( L \) is a complete lattice. Recall that compatible relations on a lattice \( L \) are subuniverses of direct powers of \( L \). Since we restrict ourselves exclusively to complete lattices, we additionally require that compatible relations are complete, i.e., closed with respect to arbitrary joins and meets. In particular, all tolerances and congruences of \( L \) are complete by definition. We are particularly interested in the following sets of relations:

- \( \text{Re}(L) \) — the set of all reflexive compatible relations on \( L \);
- \( \text{Tol}(L) \) — the set of all (compatible) tolerances on \( L \);
- \( \text{OR}_\leq(L) \) — the set of all \( \leq \)-ordered relations on \( L \) (see definition below);
- \( \text{OR}_\geq(L) \) — the set of all \( \geq \)-ordered relations on \( L \);
- \( \text{ReOR}_\leq(L) \) — the set of all reflexive \( \leq \)-ordered relations on \( L \);
- \( \text{ReOR}_\geq(L) \) — the set of all reflexive \( \geq \)-ordered relations on \( L \).

**Definition 2.1.** Following Valentini [19] we say that a binary relation \( R \) on \( L \) is \( \leq \)-**ordered** if it satisfies the following conditions:

1. given any \( u, x, y, z \in L \), if \( u \leq x \), \( (x, y) \in R \) and \( y \leq z \) then \( (u, z) \in R \);
2. given any \( t \in L \) and \( A \subseteq L \), if \( (a, t) \in R \) for every \( a \in A \) then \( (\bigvee A, t) \in R \);
3. given any \( t \in L \) and \( A \subseteq L \), if \( (t, a) \in R \) for every \( a \in A \) then \( (t, \bigwedge A) \in R \).

The \( \geq \)-**ordered** relations are defined dually.

**Remark 2.2.** Taking in Definition 2.1 \( A = \emptyset \), we get that every \( \leq \)-ordered relation contains all pairs \((0, t)\) and \((t, 1)\), \( t \in L \). Dually, every \( \geq \)-ordered relation contains all pairs \((t, 0)\) and \((1, t)\), \( t \in L \).

All the sets introduced above have a natural quantale structure with set-theoretical intersection \( \cap \) in role of \( \lor \). Let us emphasize that in this situation the inclusion relation \( \supseteq \) is in the role of the quantale’s order relation \( \leq \). In other words, smaller elements of quantale correspond to bigger subsets of \( L^2 \). Quantales with universe \( \text{OR}_\leq(L) \) were introduced in [19] by Valentini; they
have usual relational product \( \circ \) in role of multiplication. As common in universal algebra, we denote these quantales by \( \text{OR}_\leq (L) \). Note that the quantales \( \text{OR}_\leq (L) \) are unital but not integral. Their multiplicative identity element is the order relation \( \leq \) while the greatest element (the smallest as a set) is the ‘corner’ relation \( \Gamma(L) = \{(x, y) \in L^2 \mid x = 0 \text{ or } y = 1\} \). It is easy to see that \( \text{OR}_\geq (L) \) is also a unital quantale with respect to the same operations \( \cap \) and \( \circ \).

The mapping \( R \mapsto R^\circ \) establishes an anti-isomorphism between \( \text{OR}_\leq (L) \) and \( \text{OR}_\geq (L) \). In particular, the relation \( \Gamma(L)^\circ = \{(x, y) \in L^2 \mid x = 1 \text{ or } y = 0\} \) is the greatest element of \( \text{OR}_\geq (L) \).

Now we give two new alternative ways to define ordered relations which seem to be more transparent and, in particular, show that the ordered relations are compatible.

**Proposition 2.3.** Let \( R \) be a binary relation on a complete lattice \( L \). Then the following are equivalent:

1. \( R \) is a \( \leq \)-ordered relation;
2. \( R \) is the universe of a complete subdirect square of \( L \) containing the ‘corner’ element \((0,1)\); note that such subsets of \( L^2 \) were studied in [15] and denoted there by \( \text{Subd}^{\#1}(L^2) \);
3. \( R \) is a compatible binary relation on \( L \) containing \( \Gamma(L) \).

**Proof.** Let first \( R \) be \( \leq \)-ordered. Take an arbitrary family \((x_i, y_i) \in R, i \in I\), and let \( x = \bigwedge \{x_i \mid i \in I\}, \ y = \bigwedge \{y_i \mid i \in I\} \). Then, by the first condition of Definition 2.1, \((x, y_j) \in R\) for every \( j \in I \) and by the third condition, \((x, y) \in R \). Thus, \( R \) is closed with respect to arbitrary meets. That \( R \) is closed with respect to arbitrary joins, is proved similarly. Using the second and the third condition of Definition 2.1 in case \( A = 0 \), we conclude that \((a, 1), (0, a) \in R \) for every \( a \in L \). Thus, \( R \) contains the relation \( \Gamma(L) \). This completes the proof of the implication (1) \( \Rightarrow \) (3).

The implication (3) \( \Rightarrow \) (2) being trivial, we prove that (2) implies (1). Assume that \( R \) is a complete subdirect square of \( L \) and \((0,1) \in R \). Then the second and the third condition of Definition 2.1 are obviously fulfilled. Let \( u, x, y, z \in L, u \leq x, (x, y) \in R \) and \( y \leq z \). Since \( R \) is a subdirect square of \( L \), there exist \( v, w \in L \) such that \((u, v), (w, z) \in R \). Hence, \((u, 1) = (u, v) \vee (0, 1) \in R, (0, z) = (w, z) \wedge (0, 1) \in R \) and \((u, z) = ((u, 1) \wedge (x, y)) \vee (0, z) \in R \).

This completes the proof of the implication (2) \( \Rightarrow \) (1). \( \square \)

Our next aim is to show that the set \( \text{Re}(L) \) has a natural quantale structure, too, again with \( \circ \) in role of multiplication. First we present two lemmas that will be also useful when studying tolerances. The first of them is well known and needs no proof. In what follows, given a binary relation \( R \) on \( L \), we denote by \( R_\leq \) the intersection \( R \cap \leq \). The relation \( R_\geq \) is defined similarly.

**Lemma 2.4.** Let \( R \) be a reflexive compatible relation on \( L \). Then the following claims hold:

1. if \((x, z) \in R, x \leq y \leq z \) then \((x, y), (y, z) \in R\);
the following identities hold on $A$

For instance, any algebra with lattice reduct admits a majority term. An algebra $A$ admitting a majority term is called a majority algebra. The following result is implicitly contained in the main theorem of \cite{14}.

\begin{proposition}
The set of all reflexive compatible binary relations on a finite majority algebra is an integral involutive quantale with respect to set-theoretical intersection $\cap$ and relational product $\circ$.
\end{proposition}
Proof. Let $A$ be a finite majority algebra with majority term $m$. As in Proposition 2.6, only one of the distributive laws has to be proved, and due to finiteness, it suffices to prove the inclusion $(R \circ S) \cap (R \circ T) \subseteq R \circ (S \cap T)$ for arbitrary compatible reflexive binary relations $R, S, T$ of $A$. Let $(x, z) \in (R \circ S) \cap (R \circ T)$, that is, there exist $u, v \in A$ such that

$$(x, u), (x, v) \in R, \quad (u, z) \in S, \quad (v, z) \in T.$$ 

Put $y = m(u, z, v)$. Then:

$$(x, y) = (m(x, z, x), m(u, z, v)) \in R,$$ 

$$(y, z) = (m(u, z, v), m(z, z, v)) \in S,$$ 

$$(y, z) = (m(u, z, v), m(u, z, z)) \in T,$$

hence, $(x, z) \in R \circ (S \cap T)$.

Example 2.8. We present an example showing that the finiteness condition in Proposition 2.7 is essential. It also shows that there is no analog of Proposition 2.6 for non-complete setting.

Let $L$ be the interval $[0, 1]$ in $\mathbb{R}$, with the natural order relation. Clearly, $L$ is a lattice, hence a majority algebra. Of course, $L$ is even a complete lattice but here we consider it as a member of the variety of all lattices. This means that we do not require that its compatible relations be necessarily complete.

Let $R = \{(x, y) \mid 0 \leq x < y < 1\} \cup \Delta$ and $S_r = [r, 1]^2 \cup \Delta$, $r \in L^-$, where $L^- = L \setminus \{1\}$. Clearly, $R$ and all $S_r$ are compatible reflexive binary relations of $L$ but $R$ is not complete. It is easy to check that $(0, 1) \in R \circ S_r$ for any $r \in L^-$, thus $(0, 1)$ is contained in $\bigcap\{R \circ S_r \mid r \in L^-\}$. On the other hand, $\bigcap\{S_r \mid r \in L^-\} = \Delta$, hence $R \circ \bigcap\{S_r \mid r \in L^-\} = R$. Since obviously $(0, 1) \notin R$, we have $\bigcap\{R \circ S_r \mid r \in L^-\} \neq R \circ \bigcap\{S_r \mid r \in L^-\}$. We conclude that the set $\text{Re}(L)$ is not a quantale with respect to intersections and relational product operation.

Next we consider the quantale of tolerances. In view of [2], the set $\text{Tol}(L)$ forms a quantale with the set-theoretical intersection $\cap$ in role of $\vee$ and the multiplication defined as follows:

$$S \otimes T = (S \circ \geq \circ T) \cap (T \circ \leq \circ S).$$ 

We will denote this quantale by $\text{Tol}(L)$. This quantale is integral with multiplicative identity element $\Delta$. Obviously, this quantale is involutive, too.

It appeared implicitly in [16] that there is another quantale structure on $\text{Tol}(L)$, namely $(\text{Tol}(L), \cap, *)$ where the operation $*$ is defined as follows:

$$S * T = \{(x, y) \in L^2 \mid (x \lor y, x \land y) \in S \circ T\}.$$ 

The tolerance $S * T$ was called in [16] the *symmetrized product* of $S$ and $T$. Note that the facts that $*$ is associative and distributes over intersections were proved in [16] for the lattices of finite length. The proofs, however, remain valid for all complete lattices.
We show now that the operations $\otimes$ and $*$ are opposite to each other meaning that $(\text{Tol}(L), \cap, *) = \text{Tol}(L)^{\text{op}}$.

**Lemma 2.9.** Given any lattice $L$ and its tolerances $S$ and $T$, we have $S \ast T = T \otimes S$.

**Proof.** We must prove that $(x, z) \in S \ast T$ iff $(x, z) \in T \otimes S$, for every $x, z \in L$. By Lemma 2.5 it suffices to consider only the case $x \leq z$. Assume first that $(x, z) \in S \ast T$. Then there is $y \in L$ such that, $(z, y) \in S$ and $(y, x) \in T$. Since tolerances are reflexive and symmetric, we immediately have $(x, x \vee y) \in T$ and $(y \wedge z, z) \in S$. Since also $x \vee y \geq y \wedge z$, we have $(x, z) \in T \circ \geq \circ S$ and one can prove similarly $(x, z) \in S \circ \leq \circ T$. Hence, $(x, z) \in T \otimes S$.

For the converse, let $(x, z) \in T \otimes S$. Then there exist $u, v \in L$ such that $(x, u) \in T$, $u \geq v$ and $(v, z) \in S$. Without loss of generality, $u \geq x$ and $v \leq z$. Now, $(x, u) \in T$, $x \leq x \vee v \leq u$ imply $(x, x \vee v) \in T$ and $(v, z) \in S$, $v \leq x \vee v \leq z$ imply $(x \vee v, z) \in S$. Hence, $(x, z) \in T \circ S$ and $(x \vee z, x \wedge z) = (z, x) \in S \circ T$. Thus, $(x, z) \in S \ast T$. \hfill $\Box$

In conclusion of this section we consider the quantales of reflexive ordered relations $ReOR_{\leq}(L) = \text{Re}(L) \cap OR_{\leq}(L)$ and $ReOR_{\geq}(L) = \text{Re}(L) \cap OR_{\geq}(L)$.

**Proposition 2.10.** The set $ReOR_{\leq}(L)$ ($ReOR_{\geq}(L)$) is a principal filter generated by the relation $\leq$ ($\geq$) of the lattice of all compatible binary relations of the lattice $L$.

**Proof.** Let $R$ be a compatible binary relation on $L$ that contains $\leq$. Recall that by our agreement, $R$ is complete. Since $\Delta \subseteq \leq$, the relation $R$ is reflexive and since $\Gamma(L) \subseteq \leq$, it is $\leq$-ordered. Thus, $R \in ReOR_{\leq}(L)$.

For the converse, assume that $R \in ReOR_{\leq}(L)$ and let $x, y \in L$, $x \leq y$. Then by reflexivity, $(x, x) \in R$ and by definition of $\leq$-ordered relation also $(x, y) \in R$. This completes the proof. \hfill $\Box$

Since $\leq$ ($\geq$) is the multiplicative identity element of $ReOR_{\leq}(L)$ (respectively $ReOR_{\geq}(L)$) we have the following immediate corollary.

**Corollary 2.11.** The quantale $ReOR_{\leq}(L)$ ($ReOR_{\geq}(L)$) is the integral part of the quantale $OR_{\leq}(L)$ ($OR_{\geq}(L)$) and as such, is integral itself.

**Theorem 2.12.** The mappings
\[ \alpha: ReOR_{\geq}(L) \rightarrow \text{Tol}(L), \quad \alpha(R) = R \cap R^\circ; \]
and
\[ \beta: \text{Tol}(L) \rightarrow ReOR_{\geq}(L), \quad \beta(T) = \geq \circ T. \]
establish a natural isomorphism between the quantales $ReOR_{\geq}(L)$ and $\text{Tol}(L)$.

**Proof.** Obviously, if $R$ is a compatible reflexive binary relation on $L$ then $R \cap R^\circ$ is a tolerance of $L$. On the other hand, if $T$ is a tolerance of $L$ then the relation $\geq$ is contained in $\geq \circ T$, hence by Proposition 2.10 $\geq \circ T$ is a reflexive $\geq$-ordered relation on $L$. To show that the mappings $\alpha$ and $\beta$ establish a
bijection between \( \text{ReOR}_\geq(L) \) and \( \text{Tol}(L) \), it suffices to prove the following two equalities:

\[
\geq \circ (R \cap R^*) = R, \quad R \in \text{ReOR}_\geq(L); \quad (\geq \circ T) \cap (\geq \circ T)^* = T, \quad T \in \text{Tol}(L).
\]

(2.1) If \( R \in \text{ReOR}_\geq(L) \) then \( \geq \circ R = R \) and \( \geq \circ R^* = \nabla \) because \( \nabla = (\geq \circ \leq) \subseteq \geq \circ R^* \subseteq \nabla \). Hence, using Proposition 2.6, we have:

\[
\geq \circ (R \cap R^*) = \geq \circ R \cap \geq \circ R^* = \nabla \cap \nabla = \nabla
\]

which proves (2.1). As for (2.2), denote the left hand side of the formula by \( A \).

Then obviously \( T \subseteq A \) and in order to prove the converse, due to Lemma 2.5 it suffices to show that \( A \subseteq T \). Let \( (x, z) \in A, x \leq z \). Then \( (x, z) \in \geq \circ T \), that is, there exists \( y \in L \) such that \( x \geq y \) and \( (y, z) \in T \). Now, \( x \leq z \) implies \( (x, z) = (x, x) \lor (y, z) \), thus \( (x, z) \in T \).

It remains to prove that the bijection \( \beta \) is an isomorphism of quantales. Let \( T_i, i \in I \), be a family of tolerances of \( L \). Then, due to Proposition 2.6 we have the equality

\[
\geq \circ (\bigcap \{ T_i \mid i \in I \}) = \bigcap \{ \geq \circ T_i \mid i \in I \},
\]

proving that \( \beta \) preserves all intersections. Finally, if \( S, T \in \text{Tol}(L) \) then, using again Proposition 2.6, we calculate:

\[
\geq \circ (S \otimes T) = \geq \circ ((S \circ \geq \circ T) \cap (T \circ \leq \circ S))
\]

\[
= (\geq \circ S \circ \geq \circ T) \cap (\geq \circ T \circ \leq \circ S)
\]

\[
= ((\geq \circ S) \circ (\geq \circ T)) \cap \nabla
\]

\[
= (\geq \circ S) \circ (\geq \circ T).
\]

This proves that \( \beta \) preserves the multiplication, too. The mapping \( \alpha \) as the inverse of a quantale isomorphism is a quantale isomorphism, too. The proof is complete. \( \square \)

**Remark 2.13.** For some steps of the proof we could have referred to Bandelt’s paper [1] where the relationship between tolerances and ordered relations of (not necessarily complete) lattice was studied. Note that Bandelt’s \( \Gamma(L) \) is the set of all compatible binary relations on \( L \) that contain \( \leq \).

### 3. Quantales from join and meet endomorphisms

**Definition 3.1.** We call a mapping \( f : L \to L \) a join endomorphism (meet endomorphism) of \( L \) if it preserves all joins (meets). The mapping \( f \) is called decreasing (increasing) if \( f(x) \leq x \) \( (f(x) \geq x) \) for all \( x \in L \).

We denote the set of all join endomorphisms of \( L \) by \( \text{End}_\vee(L) \) and the set of all decreasing join endomorphisms of \( L \) by \( \text{End}_{\vee,\downarrow}(L) \). The sets \( \text{End}_\wedge(L) \) and \( \text{End}_{\wedge,\uparrow}(L) \) are defined dually.

It is known that the set \( \text{End}_\vee(L) \) forms a quantale with respect to the pointwise defined join and the usual composition of mappings what we will
denote just by · (see [18], page 21). This quantale End_{∨}(L) is unital but not integral. Obviously, the multiplicative identity element of End_{∨}(L) is the identity mapping $id_L$ while its greatest element is the almost constant mapping that preserves 0 and sends all nonzero elements of $L$ to 1. It is also obvious that the integral part of End_{∨}(L) is End_{∨,↓}(L), the quantale of all decreasing join-endomorphisms of $L$. By duality, the set End_{∧}(L) forms a quantale with respect to the pointwise defined meet and the usual composition of mappings. This quantale will be denoted by End_{∧}(L). Again, the quantale End_{∧}(L) is unital but not integral. Its multiplicative identity element is $id_L$ while its greatest element (in terms of the quantale $(\text{End}_{∧}(L), \land, \cdot)$ is the almost constant mapping that preserves 1 and sends all other elements of $L$ to 0. The integral part of End_{∧}(L) is End_{∧,↑}(L), the quantale of all increasing meet endomorphisms of $L$.

It is interesting to mention that in 1977 the quantale End_{∨,↓}(L) implicitly appeared in the Wille’s paper [21] on order affine completeness of finite lattices. Recall that a lattice is called order affine complete if its polynomial functions can be described as functions that preserve congruences and the order relation. Wille’s main result in [21] can be stated as follows: A finite lattice $L$ is order affine complete if and only if the quantale End_{∨,↓}(L) is generated by its multiplicative idempotents (which in fact correspond to the congruences of $L$). When preparing the book [16], the authors decided to present Wille’s result in the language of tolerances. This lead them to introducing the symmetric product of tolerances.

The join and meet endomorphisms appear naturally in residuation theory [3, 4]. This theory has been developed for (partially) ordered sets but we are interested only in the case of lattices. Let $f, f^*$ be mappings from $L$ to $L$. We say that $(f, f^*)$ is a residuated pair if

$$f(x) \leq y \iff x \leq f^*(y)$$

for every $x, y \in L$. In this situation $f$ is called a residuated mapping and $f^*$ its residual. The correspondence $f \leftrightarrow f^*$ between residuated mappings and residuals is one-to-one; it is given by the formulas:

$$f^*(y) = \bigvee \{ x \in L \mid f(x) \leq y \},$$

$$f(x) = \bigwedge \{ y \in L \mid x \leq f^*(y) \}.$$  

We denote the set of all residuated mappings (residuals) on $L$ by $\text{Res}(L)$ ($\text{Res}^*(L)$). The bijection $-^* : \text{Res}(L) \rightarrow \text{Res}^*(L)$ maps joins to meets:

$$\left( \bigvee \{ f_i \mid i \in I \} \right)^* = \bigwedge \{ f_i^* \mid i \in I \}$$  

and reverses the composition:

$$(fg)^* = g^*f^*.$$  

It is important for us that the equalities $\text{Res}(L) = \text{End}_{∨}(L)$ and $\text{Res}^*(L) = \text{End}_{∧}(L)$ hold for any complete lattice $L$ (see [3]). Now the formulas (3.1) and
(3.2) prove that the quantales \( \text{End}_\vee(L) \) and \( \text{End}_\wedge(L) \) are anti-isomorphic. Actually, the same fact can be derived from the results of Valentini’s paper [19]. Valentini proved (though, using different terminology) that the quantales \( \text{OR}_\leq(L) \) and \( \text{End}_\wedge(L) \) are isomorphic for any complete lattice \( L \). One can similarly prove that the quantales \( \text{OR}_\leq(L) \) and \( \text{End}_\vee(L) \) are anti-isomorphic. Obviously, these two facts again yield anti-isomorphism between quantales \( \text{End}_\vee(L) \) and \( \text{End}_\wedge(L) \). Perhaps it is worth to mention that the bijections between \( \text{OR}(L) = \text{Subd}^{01}(L^2) \), \( \text{End}_\vee(L) \) and \( \text{End}_\wedge(L) \) were independently also obtained in [15].

Note that Valentini’s isomorphism we mentioned above is established by the mapping \( \varphi: \text{OR}_\leq(L) \rightarrow \text{End}_\wedge(L) \), \( \varphi(R) = f_R \), where
\[
f_R(x) = \bigvee \{z \mid (z, x) \in R\}, \quad x \in L.
\]
The inverse of \( \varphi \) is given by the formula
\[
\varphi^{-1}(f) = R_f = \{(x, y) \mid x \leq f(y)\}.
\]
Similarly, the mapping \( \psi: \text{OR}_\leq(L) \rightarrow \text{End}_\vee(L) \), \( \psi(R) = f^R \), where
\[
f^R(x) = \bigwedge \{z \mid (x, z) \in R\}
\]
establishes an anti-isomorphism between \( \text{OR}_\leq(L) \) and \( \text{End}_\vee(L) \). Its inverse is
\[
\psi^{-1}(f) = R^f = \{(x, y) \mid f(x) \leq y\}.
\]
Combining \( \varphi \) and \( \psi \) with taking the converse relation, we obtain bijections
\[
\varphi^\ast: \text{OR}_\geq(L) \rightarrow \text{End}_\wedge(L), \quad \varphi^\ast(R) = \varphi(R^\ast);
\]
\[
\psi^\ast: \text{OR}_\geq(L) \rightarrow \text{End}_\vee(L), \quad \psi^\ast(R) = \psi(R^\ast).
\]
The first of these mappings is an anti-isomorphism between quantales \( \text{OR}_\geq(L) \) and \( \text{End}_\wedge(L) \), while the other is in fact an isomorphism between the quantales \( \text{OR}_\leq(L) \) and \( \text{End}_\vee(L) \).

Obviously, any isomorphism (anti-isomorphism) between unital quantales induces an isomorphism (anti-isomorphism) between their integral parts. Thus the map \( \varphi \) (\( \psi \)) restricted to \( \text{ReOR}_\leq(L) \) is an isomorphism (anti-isomorphism) between quantales \( \text{ReOR}_\leq(L) \) and \( \text{End}_\wedge,\gamma(L) \) (\( \text{ReOR}_\leq(L) \) and \( \text{End}_\vee,\downarrow(L) \)). In what follows we denote the restrictions of \( \varphi \) and \( \psi \) to \( \text{ReOR}_\leq(L) \) by the same characters \( \varphi \) and \( \psi \), respectively.

Now, having in our disposal isomorphisms \( \psi^\ast \) and \( \beta \) (introduced in Theorem 2.12), we obtain their composition
\[
\gamma = \psi^\ast \beta: \text{Tol}(L) \rightarrow \text{End}_{\vee,\downarrow}(L).
\]
Next proposition provides the formulas for calculating the values of \( \gamma \) and \( \gamma^{-1} \).

**Proposition 3.2.** Given an arbitrary complete lattice \( L \), we have:

1. \( \gamma(T) = f^T \) where \( f^T(x) = \bigwedge \{z \mid (x, z) \in T\} \) for every \( T \in \text{Tol}(L) \);
2. \( \gamma^{-1}(f) = \{(x, y) \mid f(x \vee y) \leq x \wedge y\} \) for every \( f \in \text{End}_{\vee,\downarrow}(L) \).
Proof. If \( T \in \text{Tol}(L) \) then
\[
\gamma(T) = \psi^{-1}(\geq \circ T) = \psi(T \circ \leq) = f^{T \circ \leq}.
\]
Take \( x \in L \) and denote \( a = f^{T \circ \leq}(x), \ b = f^T(x) \). Since \( T \subseteq T \circ \leq \), we immediately have \( a \leq b \). On the other hand, by definition of \( a \), the pair \((x, a)\) is contained in \( T \circ \leq \). Thus, there exists \( y \in L \) such that \((x, y) \in T \) and \( y \leq a \). Note also that \((x, x) \in T \circ \leq \), hence \( a \leq x \) by the definition of the mapping \( f^{T \circ \leq} \).

Now \((x, y) \in T\) together with inequalities \( y \leq a \leq x \) implies \((x, a) \in T\), thus \( b \leq a \). This proves the equality \( a = b \), and the first claim of the proposition.

In order to prove the second claim, we take \( f \in \text{End}_{\vee,\downarrow}(L) \) and calculate:
\[
\gamma^{-1}(f) = \beta^{-1}(\psi^{-1})^{-1}(f) = \alpha(\psi^{-1}(f)^-)
= \alpha(\{(y, x) \mid f(x) \leq y\})
= \{(x, y) \mid f(x) \leq y \text{ and } f(y) \leq x\}
= \{(x, y) \mid f(x \vee y) \leq x \wedge y\}.
\]
Note that the last equality holds due to \( f \in \text{End}_{\vee,\downarrow}(L) \). \(\Box\)

Remark 3.3. The one-to-one correspondence between tolerances and decreasing join endomorphisms given in Proposition 3.2 is not a new result. Janowitz [13] pointed out this correspondence already in 1986 and for congruences the similar correspondence was established already in 1965 [12]. In 1988 this fact was applied by Hobby and McKenzie for building up tame congruence theory [11]. Our aim was to show that Valentini’s results restricted to integral case lead to the same bijection between tolerances and decreasing join endomorphisms as that discovered by Janowitz.

4. The representation theorems

We start with an analog of Cayley’s theorem for integral quantales. First introduce special maps called translations. Given a monoid \((A, \cdot)\) and \( a \in A \), we define the mappings \( \lambda_a, \rho_a : A \to A \) by formulas \( \lambda_a(x) = ax, \ \rho_a(x) = xa \) \((x \in A)\). We call \( \lambda_a \) and \( \rho_a \) the left and right translation determined by \( a \), respectively.

**Theorem 4.1.** For any integral quantale \( Q \), the mapping \( F : Q \to \text{End}_{\vee,\downarrow}(L) \), \( F(a) = \lambda_a \), where \( L \) is the underlying lattice of \( Q \) is an embedding of quantales.

Proof. First observe that, given any \( a \in Q \), the left translation \( \lambda_a \) is a (complete) decreasing join-endomorphism of \( L \). Indeed, it follows from (1.1) that \( \lambda_a \) preserves all joins. Also, for every \( x \in Q \) we have \( ax \leq 1 \cdot x = x \), that is, \( \lambda_a \) is decreasing. That \( F \) preserves the multiplication and is one-to-one is proved exactly as in the proof of Cayley’s theorem for groups. That \( F \) preserves the joins directly follows from the distributive law \((\bigvee\{a_i \mid i \in I\})x = \bigvee\{a_ix \mid i \in I\}\) that holds for all \( a_i, x \in L, i \in I \). \(\Box\)
Corollary 4.2. For any integral quantale Q, the mapping \( G : Q \to \text{Tol}(L) \), 
\[ G(a) = \{(x, y) \mid a(x \lor y) \leq x \land y \}, a \in Q, \] 
where L is the underlying lattice of Q is an embedding of quantales.

Proof. This follows from Proposition 3.2 if we observe that \( G = \gamma^{-1}F \). □

Actually, Theorem 4.1 as well as Corollary 4.2 can be derived from Valentini’s results [19]. Let Q be a unital quantale with underlying lattice L. For any \( a \in Q \), the set \( R_a = \{(x, y) \mid ax \leq y\} \) was defined by Valentini. He proved that \( R_a \in \text{OR}_\leq(L) \) for any \( a \in Q \) and he claimed that the mapping \( H : Q \to \text{OR}_\leq(L) \), \( H(a) = R_a \), is an embedding of quantales. In fact, the latter is not exactly so. Valentini proved that H preserves the joins and that 
\[ R_{ab} = R_b \circ R_a \] 
for all \( a, b \in Q \). Thus, actually H embeds Q into \( \text{OR}_\leq(L)^{op} \) which is isomorphic to \( \text{OR}_\geq(L) \). Clearly, the mapping \( H : Q \to \text{OR}_\leq(L) \), \( \text{H}^\circ \) embeds Q into \( \text{OR}_\geq(L) \).

Let now the quantale Q be integral. Then it is easy to see that \( \text{H}(Q) \subseteq \text{ReOR}_\leq(L) \) and the composition map \( \alpha^\circ \text{H} \) embeds Q into \( \text{Tol}(L) \). We now show that this map coincides with the mapping G (from Corollary 4.2).

Proposition 4.3. For any integral quantale Q with underlying lattice L, the mappings \( \alpha^\circ \text{H} \) and \( G \) from Q to \( \text{Tol}(L) \) coincide.

Proof. Let \( a \in Q \). We must prove that \( \alpha^\circ \text{H}(a) = G(a) \), that is, for any \((x, y) \in L^2\), 
\[ (x, y) \in \alpha^\circ \text{H}(a) \iff (x, y) \in G(a). \]
The latter is equivalent to 
\[ (x, y) \in R_a \cap R_a^\circ \iff (x, y) \in G(a) \]
or, in other terms, 
\[ ax \leq y \land ay \leq x \iff a(x \lor y) \leq x \land y \]
which clearly holds due to \( a(x \lor y) = ax \lor ay, ax \leq x \) and \( ay \leq y \). □

Our next aim is to describe the image of Q in \( \text{Tol}(L) \) under the representation map G. We will show that the tolerances \( G(a) \) can be characterized in terms of a certain invariance or closure condition and also as tolerances principal in some sense. For every \( b \in Q \), the right translation \( \rho_b \) is a residuated map on L. Thus, it has a residual determined by the formula:
\[ \rho_b^\circ(y) = \bigvee \{x \in L \mid \rho_b(x) \leq y\} = \bigvee \{x \in L \mid xb \leq y\}. \]
Recall that every integral quantale has a structure of residuated lattice [7, 8] and in standard notation of the theory of residuated lattices \( \rho_b^\circ(x) = x/b \).

Let \( A = A_Q \) be the algebra \( (L, \{\rho_b \mid b \in Q\}) \), that is, A is the underlying lattice of Q equipped with new unary operations \( \rho_b, b \in Q \). Tolerances of A are the tolerances T of L such that \((x, y) \in T\) implies \((xb, yb) \in T\) for every \( b \in Q \). Let us remind that by our earlier agreement tolerances of a complete
Lemma 4.4. The set Tol(\(A\)) of all (complete) tolerances of \(A\) forms a subquantale of Tol(\(L\)).

Proof. Clearly, Tol(\(A\)) is closed with respect to all intersections. It remains to show that Tol(\(A\)) is closed with respect to \(\otimes\), or equivalently, with respect to \(\ast\). Let \(S,T \in\) Tol(\(A\)) and \((x, z) \in S \ast T\), that is, there exists \(y \in Q\) such that \((x \vee z, y) \in S\) and \((y, x \wedge z) \in T\). Since \(S\) and \(T\) are symmetric, \((x \wedge z, y) \in T\) and \((y, x \vee z) \in S\), and since \(T, S \in\) Tol(\(A\)), given any \(b \in Q\), we have \(((x \wedge z)b, yb) \in T\) and \((yb, x \vee z)b) \in S\), hence \(((x \wedge z)b, (x \vee z)b) \in T \circ S\).

Now, using \((x \wedge z)b \leq xb \wedge zb \leq xb \vee zb = (x \vee z)b\) and Lemma 2.5 we have \((xb \wedge zb, zb \vee zb) \in T \circ S\), hence \((xb \vee zb, xb \wedge zb) \in S \circ T\), implying \((xb, zb) \in S \ast T\).

As we know, there is a quantale isomorphism \(\gamma: Tol(L) \rightarrow End_{\vee, \otimes}(L)\). Next lemma shows which decreasing join endomorphisms of the lattice \(L\) belong to \(\gamma(\text{Tol}(A))\).

Lemma 4.5. Let \(Q\) be an integral quantale with underlying lattice \(L\), \(T \in\) Tol(\(L\)) and \(\gamma(T) = f\). Then \(T \in\) Tol(\(A\)) if and only if \(f(xy) \leq f(xy)y\) for all \(x, y \in Q\).

Proof. Assume first that \(T \in\) Tol(\(A\)). It is easy to see that \((f(x), x) \in T\) and, by our assumption, \((f(xy), xy) \in T\) for all \(x, y \in Q\). Now \(f(xy) \leq f(xy)y\) follows from Proposition 3.2 (1) and \(f = \gamma(T)\).

Assume now that \(f(xy) \leq f(xy)y\) for all \(x, y \in Q\). Let \((a, b) \in T\); we have to show that \((ac, bc) \in T\) for every \(c \in Q\). Let first \(a \leq b\). Then clearly \(ac \leq bc\) and by the definition of \(f = \gamma(T)\), from \((a, b) \in T\) it follows \(f(b) \leq a\), hence also \(f(b)c \leq ac\). Now, using our assumption, we obtain \(f(bc) \leq f(bc)c \leq ac \leq bc\). Since \((f(bc), bc) \in T\), the latter implies \((ac, bc) \in T\).

If \((a, b) \in T\) is arbitrary, we put \(d = a \wedge b\). Then \((d, a), (d, b) \in T\) and by what we just proved, \((dc, ac), (dc, bc) \in T\). But then

\[(ac, bc) = (dc, bc) \vee (ac, dc) \in T\].

For any \(a, b \in Q\), we denote by \(T_A(a, b)\) the principal tolerance of \(A\) generated by the pair \((a, b)\). Thus, \(T_A(a, b)\) is the least tolerance of \(A\) that contains \((a, b)\).

Theorem 4.6. Let \(Q\) be an integral quantale and \(G\) the representation map from Corollary 4.2. Then the following are equivalent for every \(T \in\) Tol(\(L\)):

1. there exists \(a \in Q\) such that \(T = G(a)\);
2. \(T\) is invariant with respect to all right translations of \(Q\), and their residuals, that is, for any \((x, y) \in T\) and \(b \in Q\), also \((xb, yb), (x/b, y/b) \in T\);
3. there exists \(a \in Q\) such that \(T = T_A(a, 1)\).
In fact, \( a = \bigwedge \{ z \in Q \mid (z, 1) \in T \} \) and \( G(a) = T_A(a, 1) \).

Proof. We first show that the first two claims are equivalent and then prove the equality \( G(a) = T_A(a, 1) \).

Denote \( T_a = G(a) = \{ (x, y) \in Q^2 \mid a(x \vee y) \leq x \land y \} \) and prove that it is invariant under the mappings \( \rho \) and \( \rho^* \), for every \( b \in Q \). Since \( \lambda_a \in \text{End}_{\vee, \wedge}(L) \), it suffices to prove that \( ax \leq y \) implies both 1) \( a(xb) \leq yb \) and 2) \( a(x/b) \leq y/b \) for arbitrary \( x, y \in Q \). The first of them is easy because \( a(xb) = (ax)b \leq yb \). To prove the second claim, we have to show that if \( ax \leq y \) and \( zb \leq x \) then \( az \leq y/b \). But indeed, \( y/b \) is the join of all such \( w \in Q \) that \( wb \leq y \). Since \( (az)b = a(zb) \leq ax \leq y \), the element \( az \) is one of such \( w \). Thus, \( az \leq y/b \) and we have proved that (1) implies (2).

Let now \( T \in \text{Tol}(L) \), and assume that \( T \) is closed under all right translations of \( Q \) and their residuals. Let \( a = \bigwedge \{ x \in L \mid (x, 1) \in T \} \). We are going to prove that \( T = T_a \). Let \( (x, y) \in Q^2 \), we must prove that \( (x, y) \in T \) if and only if \( (x, y) \in T_a \). By Lemma 2.5, we may assume that \( x \leq y \).

Assume first that \( (x, y) \in T \). Since \( T \) is invariant under all residual mappings \( \rho^*_y \), we have \( (x/y, 1) = (x/y, y/y) \in T \), hence, \( a \leq x/y \) implying \( ay \leq x \). But then clearly \( a(x \lor y) \leq x \land y \), that is, \( (x, y) \in T_a \).

Let now \( (x, y) \in T_a \), hence \( ay \leq x \). Since \( (a, 1) \in T \) and \( T \) is closed with respect to right translations of \( Q \), we have \( (ay, y) \in T \). Now the inequalities \( ay \leq x \leq y \) imply \( (x, y) \in T \).

It remains to prove the equality \( T_a = T_A(a, 1) \). Since \( T_a \in \text{Tol}(A) \) and \( (a, 1) \in T_a \), the inclusion \( T_A(a, 1) \subseteq T_a \) is obvious. To complete the proof, it suffices to show that \( T_a \) is contained in any \( T \in \text{Tol}(A) \) such that \( (a, 1) \in T \). The proof is based on the one-to-one correspondence between tolerances and decreasing join endomorphisms given in Section 3. Let \( \gamma(T) = f \). Since \( \gamma(T_a) = \lambda_a \), the inclusion \( T_a \subseteq T \) is equivalent to \( f \leq \lambda_a \). We first observe that \( (a, 1) \in T \) yields \( f(1) \leq a \). Now, using Lemma 4.5, we have:

\[
 f(x) = f(1 \cdot x) \leq f(1)x \leq ax = \lambda_a(x).
\]

This completes the proof. \( \square \)

Remark 4.7. Valentini [19] introduced what he called right ordered relations and proved that \( R \in \text{OR}_\leq(L) \) has the form \( H(a) \) for some \( a \in L \) if and only if it is right ordered. The equivalence of the first two conditions in Theorem 4.6 could have been derived from that Valentini’s result but we preferred a direct, self-contained proof.

We have showed that for every integral quantale \( Q \) there exists a natural embedding \( G : Q \to \text{Tol}(L) \) where \( L \) is the underlying lattice of \( Q \). Moreover, we have seen that \( G(Q) \) is contained in the subquantale \( \text{Tol}(A) \) of \( \text{Tol}(L) \). It makes sense to hope that \( \text{Tol}(A) \), being “closer” to \( Q \), carries more information about \( Q \) than \( \text{Tol}(L) \). The following example and theorem confirm that expectation.
Example 4.8. We construct a quantale $Q$ such that the embedding $G$ is not a lattice homomorphism. Let $L$ be a free bounded distributive lattice in two generators $b$ and $c$. We denote $a = b \land c$ and $d = b \lor c$. Thus, $L$ is the lattice with universe $L = \{0, a, b, c, d, 1\}$ depicted in Figure 1. The commutative multiplication on $L$ is also given by the table in Figure 1. Observe that $1$ is its identity element and $0$ is the multiplicative zero. It is easy to check that this operation is associative. Indeed, clearly $(xy)z = x(yz)$ holds if $1 \in \{x, y, z\}$ and if $1 \notin \{x, y, z\}$ then $(xy)z = 0 = x(yz)$ easily follows from the multiplication table.

![Figure 1. The constructed quantale $Q$](image.png)

Figure 1. The constructed quantale $Q$

It is also an easy exercise to check that all left translations $\lambda_x, x \in L$, are join endomorphisms of $L$. Thus, $Q = (L, \lor, \cdot)$ is a commutative integral quantale.

We will show that the join of $T_b$ and $T_c$ in $\text{Tol}(L)$ is a proper subset of $T_a$. We first observe that $T = [0, a]^2 \cup [a, 1]^2$ is a tolerance of $L$. Indeed, $T$ is a subuniverse of $L$ because $x \leq y$ holds for every $x \in [0, a]$ and $y \in [a, 1]$. Moreover, $L = [0, a] \cup [a, 1]$ implies that $T$ is reflexive. Since $(0, a), (a, 1) \in T_a$, we conclude that $T \subseteq T_a$. However, $T \neq T_a$ because $(0, d) \notin T_a \setminus T$.

It remains to show that both $T_b$ and $T_c$ are contained in $T$. The proofs of these two inclusions are similar, thus it suffices to prove that $T_b \subseteq T$. Actually, any of these two inclusions follows from the other because the selfmap of $L$ that interchanges $b$ and $c$ and fixes the other elements of $L$ is the automorphism of $L$. Let $(x, y) \in T_b$. If $x, y \leq a$ then clearly $(x, y) \in T$. If $x > a$ then $(x, y) \in T_b$ implies $y \geq a$, thus again $(x, y) \in T$.

Theorem 4.9. For every integral quantale $Q$, the mapping $G : Q \to \text{Tol}(A)$ is a lattice embedding.

Proof. Recall that $Q = (Q, \lor, \cdot)$ and $\text{Tol}(A) = (\text{Tol}(A), \cap, \otimes)$. Thus, in particular, $G$ maps all joins in $L$ to corresponding intersections in $\text{Tol}(A)$. Hence, we must prove that $G$ maps all meets in $L$ to corresponding joins in $\text{Tol}(A)$.
Denoting the join operation in \( \text{Tol}(A) \) by \( \sqcup \), we must prove the equality
\[
G(\bigwedge \{a_i \mid i \in I\}) = \bigsqcup \{G(a_i) \mid i \in I\}
\]
or, equivalently,
\[
T_{\bigwedge \{a_i \mid i \in I\}} = \bigsqcup \{T_{a_i} \mid i \in I\}, \tag{4.1}
\]
for any family of elements \( a_i \in Q, i \in I \). Since \( \gamma \) is order reversing \((x \leq y \text{ in } Q \implies T_x \supseteq T_y)\), the inclusion \( \supseteq \) in (4.1) is obvious. For proving the converse, observe that
\[
(a_j, 1) \in T_{a_j} \subseteq \bigsqcup \{T_{a_i} \mid i \in I\}
\]
for every \( j \in I \), hence
\[
(\bigwedge \{a_i \mid i \in I\}, 1) = \bigwedge \{(a_i, 1) \mid i \in I\}) \in \bigsqcup \{T_{a_i} \mid i \in I\}.
\]
Finally, Theorem 4.6 yields the inclusion \( \subseteq \) in (4.1).

An important class of integral quantales is that of frames, that is, quantales in which the identity \( xy = x \land y \) holds. It is well known that a complete lattice \( L \) is the underlying lattice of a frame if and only if it is relatively pseudocomplemented, that is, for every \( a, b \in L \) there exists an element \( c \in L \) which is greatest with respect to \( a \land c \leq b \). Now Theorem 4.9 yields the following corollary.

**Corollary 4.10.** If \( L \) is a complete relatively pseudocomplemented lattice then the mapping \( G \) embeds \( L \) into the lattice of (complete) congruences of \( L \).

**Proof.** Let \( Q \) be the quantale (frame) related to \( L \) and \( A = A_Q \). By Theorem 4.9, \( G \) embeds \( L \) into the lattice \( \text{Tol}(A) \). Since the multiplication in \( Q \) coincides with meet operation, \( \text{Tol}(A) = \text{Tol}(L) \). It remains to show that all tolerances \( T \in G(L) \) are actually congruences of the lattice \( L \). Let \( T = G(a) \) where \( a \in L \). Then \( \gamma(T) = \lambda_a \), that is, under the bijection \( \gamma \), the tolerance \( T \) corresponds to the decreasing join-endomorphism \( \lambda_a \). However, in case of frames the left translation \( \lambda_a \) is idempotent which implies that the corresponding tolerance \( T \) must be a congruence (see [12]).

We conclude the section with a somewhat surprising result. We have proved above (Proposition 2.7) that any finite majority algebra \( A \) gives rise to an integral involutive quantale \( \text{Re}(A) \). It will turn out that actually all finite integral involutive quantales can be obtained in this way. For that we need some terminology from the theory of semirings.

**Definition 4.11.** An algebra \( S = (S, \lor, \land, \cdot, 0, 1) \) is called a lattice ordered semiring if \((S, \lor, \land)\) is a lattice with the smallest element 0, \((S, \cdot, 1, 0)\) is a monoid with zero element 0, multiplication distributes over joins and \( xy \leq x \land y \) holds for all \( x, y \in S \). A lattice ordered semiring \( S \) is called involutive if there exists an involution \( -^{-1} \) of its multiplicative monoid such that \((x \lor y)^{-1} = x^{-1} \lor y^{-1}\) holds for all \( x, y \in Q \).


Remark 4.12. A good source for semirings is Golan’s book [9]. Involution lattice ordered semirings appeared in [14]. Though explicit definition is not given there, it is clear from the context that they were understood precisely in the sense of the present Definition 4.11.

Obviously, every finite lattice ordered semiring is a unital quantale. On the other hand, every integral quantale $Q$ has a natural structure of lattice ordered semiring. Indeed, $x \leq 1$ for every $x \in Q$ implies $xy \leq x \land y$ for all $x, y \in Q$. Clearly, an involution on a lattice ordered semiring is an involution on a corresponding quantale and vice versa.

Theorem 4.13. Given any finite integral involutive quantale $Q$, there exists a finite majority algebra $A$ such that $Q \cong \text{Re}(A)$, the quantale of all reflexive compatible binary relations on $A$.

Proof. As we mentioned above, there is a natural structure of involutive lattice ordered semiring on $Q$; denote it by $Q$. Now, by Theorem 5.1 of [14], there exists a finite diagonal majority algebra $A$ such that $Q$ is isomorphic to the lattice ordered semiring $(S_2(A), \cap, \lor, \circ, \nabla, \Delta)$. Here $S_2(A)$ is the set of all compatible binary relations on $A$ and diagonality of $A$ means that all members of $S_2(A)$ are reflexive. Thus, $S_2(A) = \text{Re}(A)$. Clearly, the isomorphism between lattice ordered semirings is an isomorphism between related quantales. This means that $Q \cong \text{Re}(A)$ and we are done.

5. Properties of the underlying lattice

Given any complete lattice $L$, we can turn it into a quantale by defining the multiplication trivially, say, $xy = 0$ for all $x, y \in L$. If we want the quantale to be unital or even integral, the situation becomes much complicated. In [10] Grätzer and Schmidt studied the lattice of join endomorphisms of a lattice, thus, in case of a finite lattice $L$ they studied the underlying lattice of the unital quantale $\text{End}_\lor(L)$. They proved that that lattice is distributive if and only if $L$ is distributive. On the other hand, they established that in case of finite non-distributive lattice $L$ the underlying lattice of $\text{End}_\lor(L)$ is even not semimodular. Note that clearly every finite distributive lattice $L$ being relatively pseudocomplemented is the underlying lattice of the frame $(L, \lor, \land)$. In what follows, we will derive some necessary conditions for a finite lattice to be the underlying lattice of some integral quantale. First, we recall some notions.

Definition 5.1. Let $L$ be a lattice with 0. A pair of elements $a, b \in L$ is called:

- disjoint if $a \land b = 0$;
- distributive if $c \land (a \lor b) = (c \land a) \lor (c \land b)$ holds for any $c \in L$.

The lattice $L$ is called...
• pseudocomplemented if for each \( x \in L \) there exists an element \( x^* \in L \) (the pseudocomplement of \( x \)) such that for any \( y \in L \), \( y \land x = 0 \iff y \leq x^* \);

• distributive in 0 if all disjoint pairs of \( L \) are distributive.

Corresponding dual notions are defined for lattices with 1.

**Remark 5.2.** The class of pseudocomplemented lattices contains all relatively pseudocomplemented lattices but the converse is not true. The distributivity in 0 was introduced in [5] under the name “super-0-distributivity”. The latter term is somehow misleading because already in 1968 Varlet [20] has introduced 0-distributive lattices. In principle super-0-distributivity should be stronger notion than 0-distributivity but in fact our distributivity in 0 is incomparable with Varlet’s 0-distributivity. However, every distributive in 0 lattice is 0-modal in the sense that it does not contain a pentagon sublattice containing the 0.

We need the following simple lemma.

**Lemma 5.3.** Let \( L_1 \) be a sublattice of a lattice \( L \) and assume that \( L \) and \( L_1 \) have common 0. If \( L \) is distributive in 0 then \( L_1 \) is distributive in 0, too. If \( L \) is complete and pseudocomplemented and \( L_1 \) is a complete sublattice of \( L \) then \( L_1 \) is pseudocomplemented, too.

**Proof.** The first claim is clear. To prove the second one, let \( x \in L_1 \) and \( x^* \) be its pseudocomplement in \( L \). Put \( z = \bigvee \{ y \in L_1 \mid y \leq x^* \} \). As \( L_1 \) is complete, \( z \in L_1 \) and it is easy to check that \( z \) is the pseudocomplement of \( x \) in \( L_1 \). \( \square \)

In what follows we need the following facts that were proved in [5, 6].

**Theorem 5.4.** The lattice of all tolerances of any majority algebra \( A \) is pseudocomplemented and distributive in 0.

Now we are ready to state and prove the main result of this section.

**Theorem 5.5.** The underlying lattice \( L \) of any finite integral quantale is dually pseudocomplemented and distributive in 1.

**Proof.** Let \( Q \) be a finite integral quantale, \( L \) its underlying lattice and \( A = A_Q \). Since \( L \) is finite, the algebra \( A \) is a finite majority algebra. Therefore, any tolerance of \( A \) is complete, i.e., \( \text{Tol}(A) \) consists of all tolerances of the algebra \( A \). Hence, by Theorem 5.4, the underlying lattice of the quantale \( \text{Tol}(A) \) is pseudocomplemented and distributive in 0. In view of Proposition 4.9, the mapping \( \gamma \) embeds \( L = (L, \lor, \land) \) into the lattice \( (\text{Tol}(A), \cap, \sqcup) \), in particular, \( \gamma(0) = \nabla \). Now our claim follows from Lemma 5.3, because \( \nabla \) is the greatest element of \( \text{Tol}(A) \). \( \square \)

**Acknowledgment.** We thank the anonymous referee for the most valuable suggestions that helped us considerably to improve the final version of the paper.
REFERENCES


KALLE KAARLI
University of Tartu
e-mail: kaarli@ut.ee
URL: https://www.ut.ee/en/kalle-kaarli

SÁNDOR RADELECZKI
University of Miskolc
e-mail: matradi@uni-miskolc.hu
URL: https://www.uni-miskolc.hu/~matradi