Direct Product of $\ell$-algebras and Unification.
An Application to Residuated Lattices

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We describe classes of $\ell$-algebras (which are based on lattices) such that their finitely presented projective algebras are closed under finite direct product, that is, for which unification is filtering. This implies that unification in such classes is either unitary or nullary. Following ideas of S. Ghilardi [12], [14] we attempt to describe filtering unification in a variety by means of properties of factor-congruences of algebras of the variety. The results subsume some previous results, but not those of [14], and open new areas for applications like residuated lattices. In particular we show that filtering unification depends on the monoid operation, that is, unification is filtering in varieties generated by residuated lattices without zero divisors. This implies that unification in strict fuzzy logics like SMTL, IIMTL and many others is unitary or nullary.

Key words: unification, $\ell$-algebras, finitely presented algebra, projective algebra, 1-regular variety, congruence kernel, central element, factor congruence, residuated lattice, zero divisors.

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1 INTRODUCTION

In this paper we provide algebraic conditions for filtering unification in a class of lattice-based algebras, and hence, for the fact that unification type in the class is either unitary or nullary.

Unification in equational theories, called $E$-unification, is an important tool in equational reasoning applied in automated theorem proving, in particular in Automated Deduction and Term Rewriting, see [2], as well as in recognizing admissibility of inference rules in logic.

Given an equational theory $E$, a pair of terms $(t_1, t_2)$ is called a “unification problem” and a solution to it, or a unifier for it in $E$, is a substitution $\sigma$ such that $\vdash_E \sigma(t_1) = \sigma(t_2)$. The terms $t_1, t_2$ are called unifiable if there is a unifier for them. A substitution $\sigma$ is more general than a substitution $\tau$, $\tau \leq \sigma$, if there is a substitution $\theta$ such that $\vdash_E \theta \circ \sigma \approx \tau$.\(^*\)

A most general unifier, in short a mgu, for $(t_1, t_2)$, is a unifier that is more general than any unifier for $(t_1, t_2)$. A mgu can be interpreted as “the best” solution to the unification problem. An equational theory $E$ has unitary unification, or type 1 unification, if for every unifiable terms $t_1, t_2$ there is a mgu for $t_1, t_2$ in $E$. Unification types can be unitary, finitary, infinitary or nullary depending on number (in the ‘worst’ case of pairs of terms) of maximal with respect to $\leq$ unifiers, cf. [2], [12]. A theory $E$ has nullary unification, or type 0, if there are unifiable terms $t_1, t_2$, such that for any unifier $\tau$ there is another unifier which is (essentially) more general then $\tau$, i.e. a maximal unifier for unifiable terms $t_1, t_2$ does not exist. Instead of an equational theory $E$ one considers, equivalently, a corresponding equational class, a variety, $\mathcal{V}$ of algebras. For example, unification in Boolean algebras is unitary. Establishing the unification type of a given variety or a logic is often a hard problem and there are few general results like, for example S. Burris [3]: all discriminator varieties have unitary unification. Also characterizing theories, or logics, with unitary (or finitary), unification is quite hard (see e.g. [1]). However S. Ghilardi and L. Sacchetti [14] managed to characterize modal logics in which unification is filtering.

Unification in a variety $\mathcal{V}$ is called filtering if, for any pair of unifiers for given terms $t, s$ there is a unifier for $t, s$ which is more general than both of them, cf. [14]. If unification in $\mathcal{V}$ is filtering then it is either unitary or nullary. S. Ghilardi and L. Sacchetti [14] characterized normal modal logics extending K4 with filtering unification by showing that for a variety $\mathcal{V}$ of K4-algebras, the following conditions are equivalent:

\(^*\) Some authors use the reverse order and write $\sigma \leq \tau$ in this case.
(i) unification in $\mathcal{V}$ is filtering,
(ii) finitely presented projective $\mathcal{V}$-algebras are closed under binary direct products,
(iii) $\lozenge^+ \Box^+ x \rightarrow \Box^+ \lozenge^+ x = 1$ holds in $\mathcal{V}$.

Equivalence of (i) and (ii) is a more general algebraic fact that does not depend on modal logic. We will investigate varieties of $\ell$-algebras in which the condition (ii) holds. More exactly, we follow some ideas from S. Ghilardi and L. Sacchetti [14], but instead of modal K4-algebras which are not $\ell$-algebras we search for classes of lattice-based $\ell$-algebras in which the above condition (ii) holds. Our aim is to find conditions on factor-congruences of algebras from a variety $\mathcal{V}$ of lattice based algebras of the form $\mathcal{A} = (A, \wedge, \vee, 0, 1, F)$, where $F$ is a set of compatible operations, and $(A, \wedge, \vee, 0, 1)$ is a bounded lattice, such that finitely presented projective $\mathcal{V}$-algebras are closed under binary direct product. As a corollary we get that unification in $\mathcal{V}$ is filtering, which means that unification is either unitary or nullary. Note, however, that K4-algebras are not $\ell$-algebras, hence S. Ghilardi and L. Sacchetti result [14] does not follow from the results of this paper.

As an application we get that if the Stone identity $\neg\neg x \vee \neg x = 1$ holds in a variety of residuated lattices, then unification in it is filtering. This algebraic description of lattice based algebras with filtering unification gives some purely algebraic insight into reasons for a particular unification type.

After preliminaries in Section 2 that include notions of the center of a lattice, $\ell$-algebras, 1-regular algebras etc., the main result of the paper is achieved in two steps. The first step, in Section 3, is concerned with finding algebraic conditions for a variety $\mathcal{V}$ of $\ell$-algebras such that the direct product of two finitely presented $\mathcal{V}$-algebras is finitely presented. In the second step, in Section 4, extra conditions are added on a variety $\mathcal{V}$ such that the direct product of two finitely presented projective $\mathcal{V}$-algebras is again projective.

Since the main result holds for any variety $\mathcal{V}$ of lattice based $\ell$-algebras the present approach opens new areas of applications. In Section 5, we show for a variety $\mathcal{V}$ of bounded commutative integral residuated (even non-distributive or non-modular) lattices, if subdirectly irreducible residuated lattices generating the variety $\mathcal{V}$ have no zero divisors, or the Stone identity $\neg\neg x \vee \neg x = 1$ holds in $\mathcal{V}$, then unification in $\mathcal{V}$ is filtering. As a partial converse, if unification in a variety $\mathcal{V}$ of such residuated lattices is filtering and the identity $\neg x \circ \neg x = \neg x$ holds in $\mathcal{V}$, then the Stone identity holds in $\mathcal{V}$.\footnote{In the context of (ii) note that closure of finitely generated free algebras under binary products is sufficient for filtering unification but is not necessary for it, see [14]}

\footnote{Example A shows that $\neg x \circ \neg x = \neg x$ does not imply the Stone identity, hence filtering}
fication in varieties of such residuated lattices validating the Stone identity is unitary or nullary. This helps to find the unification type of several fuzzy logics. In particular this applies to varieties of SMTL-algebras, IIMTL-algebras (and include Product algebras and Gödel algebras), to some Archimedean classes of residuated chains, see [16], and others.

2 PRELIMINARIES

Finding the unification type of a given equational theory, or a variety, is often a difficult task. S. Ghilardi in [12] introduced algebraic and categorial approach to unification that can make this task much easier. The main feature of Ghilardi’s approach is that unification depends only on finitely presented and projective objects. We recall basic notions of this approach.

Given an equational theory $E$, let $\mathcal{V}_E$ be the variety corresponding to $E$ (we often write $\mathcal{V}$). An algebra $\mathfrak{B} \in \mathcal{V}_E$ is finitely presented if it is isomorphic to a finitely generated free algebra factorized by a finitely generated (or a compact) congruence; i.e. if there is a finite set of variables, $\{x_1, \ldots, x_k\} = \underline{x}$ and a finite set $S$ of equations of terms with variables in $\underline{x}$ such that $\mathfrak{B}$ is isomorphic to a quotient algebra $F_E(\underline{x})/\sim$, where $\sim$ is a (compact) congruence defined as follows:

$$t_1 \sim t_2 \text{ iff } S \vdash_E t_1 \approx t_2$$

Thus $\sim$ is the congruence generated by the set of pairs in $S$. The pair $(\underline{x}, S)$ is then called a presentation for $\mathfrak{B}$. Note that $\vdash_E$ is substitution invariant but it is not allowed to produce substitution consequences.

An algebra $\mathfrak{P}$ in a variety $\mathcal{V}$ is projective in $\mathcal{V}$ if for every $\mathfrak{A}$ and $\mathfrak{B}$ of $\mathcal{V}$ and homomorphisms $f : \mathfrak{P} \to \mathfrak{B}$, $g : \mathfrak{A} \to \mathfrak{B}$ (where $g$ is regular epi; here in $\mathcal{V}_E$ it is a surjective homomorphism) there is a homomorphism $h : \mathfrak{P} \to \mathfrak{A}$ such that the following diagram commutes:

$$\begin{array}{ccc}
\mathfrak{P} & \xrightarrow{f} & \mathfrak{B} \\
\downarrow{h} & & \downarrow{g} \\
\mathfrak{A} & \xrightarrow{g} & \mathfrak{P}
\end{array}$$

It is known, see e.g. [12], [14], that for a finitely presented algebra $\mathfrak{P} \in \mathcal{V}_E$ the following are equivalent:

unification is essential here
(i) \( \mathcal{P} \) is projective in \( \mathcal{V} \)
(ii) \( \mathcal{P} \) is a retract of a free algebra \( \mathcal{F}_E(\mathcal{X}) \) for a finite set \( \mathcal{X} \) in \( \mathcal{V} \), i.e. there are morphisms \( q \) and \( m \) and a free algebra \( \mathcal{F}_V(\mathcal{X}) \) such that \( q \circ m = 1 \) (\( q \) is onto, \( m \) is one–to–one).

\[
\begin{array}{c}
\mathcal{P} \\
m \\
\mathcal{F}_V(\mathcal{X}) \\
q \\
\mathcal{P}
\end{array}
\]

In the approach of Ghilardi [12] an \( E \)-unification problem corresponds to a finitely presented algebra \( \mathfrak{A} \in \mathcal{V}_E \). A unifier (a solution) for \( \mathfrak{A} \) is a pair given by: a projective algebra \( \mathcal{P} \) and a homomorphism \( u : \mathfrak{A} \to \mathcal{P} \).

A finitely presented algebra \( \mathfrak{A} \) is unifiable if it has a unifier. Given two unifiers \( u_1 \) and \( u_2 \) for \( \mathfrak{A} \), \( u_1 : \mathfrak{A} \to \mathcal{P}_1 \) is a more general algebraic unifier than \( u_2 : \mathfrak{A} \to \mathcal{P}_2 \), \( u_2 \preceq u_1 \), if there is a homomorphism \( g \) such that the following diagram commutes:

\[
\begin{array}{ccc}
\mathfrak{A} & \xrightarrow{u_1} & \mathcal{P}_1 \\
\downarrow \quad u_2 & & \downarrow \quad g \\
\mathcal{P}_2 & & \mathcal{P}_1
\end{array}
\]

After finding, in Section 3, conditions for a variety \( \mathcal{V} \) implying that finitely presented projective \( \mathcal{V} \)-algebras are closed under (binary) direct product we make use of the following purely algebraic result of Ghilardi and Sacchetti [14], Theorem 3.2.

**Theorem 1** (see [14]). Unification in \( \mathcal{V} \) is filtering iff finitely presented projective \( \mathcal{V} \)-algebras are closed under (binary) direct product.

To give an idea of the proof, assume that finitely presented projective \( \mathcal{V} \)-algebras are closed under finite direct product. Then, for any two unifiers for \( \mathfrak{A} \), \( u : \mathfrak{A} \to \mathcal{P}_1 \) and \( v : \mathfrak{A} \to \mathcal{P}_2 \), the unifier given by the direct product \( f : \mathfrak{A} \to \mathcal{P}_1 \times \mathcal{P}_2 \), is more general than \( u, v \), as shown in the following diagram.
Let $L$ be a bounded lattice and let $0$ and $1$ stand for the least and the greatest element of $L$. We denote the principal filter of an element $a \in L$ by $[a]$. An element $a \in L$ is called a central element of $L$ if $a$ is complemented and for all $x, y \in L$ the sublattice generated by $\{a, x, y\}$ is distributive. The central elements of $L$ form a Boolean sublattice of the lattice $L$ denoted by $\text{Cen}(L)$. Hence any element $c \in \text{Cen}(L)$ has a single complement $c$, the pair $\{c, c\}$ is also called a central pair of $L$. Any $c \in \text{Cen}(L)$ induces a congruence $\theta_c = \{(x, y) \mid x \land c = y \land c\}$.

The best known examples of $\ell$-algebras are bounded lattices, p-algebras, ortholattices and Heyting algebras (see e.g. [18] and [22]). On the other hand, S4-modal and K4-modal algebras $(A, \land, \lor, \rightarrow, \Box, \Diamond, 0, 1)$ are not $\ell$-algebras. Since $\ell$-algebras have a lattice reduct, they are congruence distributive, and the factor congruences of an $\ell$-algebra and of its underlying lattice $L$ coincide. An $\ell$-algebra will be called distributive, if its underlying lattice is distributive.
A congruence \( \theta \) of \( \text{Con}A \) is said to be \textit{compact} if for every (nonempty) subset \( \Phi \) of \( \text{Con}A \), \( \theta \leq \bigvee \Phi \) implies \( \theta \leq \bigvee F \) for some finite nonempty \( F \subseteq \Phi \).

Observe, that this definition and (1) yields \( \theta \) some \( n \phi \) of \( \text{Con}A \) a congruence class \( \theta \) (see [4] or [6]), if \( \theta = \theta(a_1, b_1) \lor \cdots \lor \theta(a_n, b_n) \), for some \( n \in \mathbb{N} \) and \( (a_1, b_1), \ldots, (a_n, b_n) \in \theta \).

An algebra \( A = (A, F) \) with a constant 1 is called 1-regular (or weakly regular) (see [4] or [6]), if \( [1]_{\varphi} = [1]_{\theta} \) implies \( \varphi = \theta \), for each \( \varphi, \theta \in \text{Con}A \).

A variety \( V \) with a constant 1 is 1-regular if each \( A \in V \) is 1-regular. Clearly, any congruence \( \varphi \) of a 1-regular algebra \( A \) is generated by its congruence class \([1]_{\varphi}\), i.e. \( \varphi = \theta([1]_{\varphi}) \). From B. Csákány characterization of regular varieties [4] it follows that

\textbf{Theorem 2.} A variety \( V \) with a constant 1 is 1-regular if and only if there exist binary terms \( d_1(x, y), \ldots, d_n(x, y) \) such that
\[
d_i(x, x) = 1, \text{ for } i = 1, \ldots, n, \text{ and } d_1(x, y) = 1, \ldots, d_n(x, y) = 1 \text{ imply } x = y.
\]

Hence in a 1-regular variety we have
\[
d_1(x, y) = 1, \ldots, d_n(x, y) = 1 \iff x = y.
\]

\textbf{Corollary 3.} Let \( V \) be a variety with a constant 1 which is 1-regular. Then

1. Every equation on terms \( t_1 = t_2 \) in \( V \) can be replaced by an equivalent finite set of equations of the form \( t = 1 \)

2. For any finitely presented algebra in \( V \) any equation in the set \( S \) of its presentations \( (x, S) \) can be taken to be an equation of the form \( t = 1 \).

3. If \( V \) has a meet-operation \( \land \), then all finitely generated congruences of its algebras \( A \) are principal of a specific form: \( \theta(s, 1) \).

\textit{Proof.} (1) and (2) are clear. In case of (3) by defining a binary term \( d(x, y) = d_1(x, y) \land \cdots \land d_n(x, y) \), in view of Theorem 2 we obtain \( d(x, y) = 1 \iff x = y \). As any finitely generated congruence \( \theta \in \text{Con}A \) has the form \( \theta = \theta([s_1 = t_1, \ldots, s_n = t_n]) \), we obtain \( \theta = \theta(d(s_1, t_1) \land \cdots \land d(s_n, t_n), 1) \). \( \square \)

It is also clear, that by adding new operations to the operations of an algebra \( A \) from a 1-regular variety, we obtain again a 1-regular algebra \( A' \).

Next, let \( (L, \land, \lor, 1) \) be a lattice with the greatest element 1, and \( A = (L, \land, \lor, 1, F) \) be an algebra. Since any \( \theta \in \text{Con}A \) is also a congruence of \( (L, \land, \lor) \), it follows that the congruence class \([1]_{\theta}\) is a filter of \( L \), called the \textit{kernel} of \( \theta \). We will denote it by \( \ker(\theta) \). If \( A \) is 1-regular, then obviously, \( \theta_1 \leq \theta_2 \iff \ker(\theta_1) \subseteq \ker(\theta_2) \), for any \( \theta_1, \theta_2 \in \text{Con}A \).
Lemma 4. Let \( A = (L, \land, \lor, 0, 1, F) \) be a 1-regular algebra. Then any compact congruence \( \theta \in \text{Con}A \) is generated by a principal filter of \( L \) and for any elements \( x, a, b \in L \) we have

\[
\theta([x]) = \theta(x, 1),
\]

(2)

\[
\theta(a, 1) \lor \theta(b, 1) = \theta(a \land b, 1).
\]

(3)

Proof. \( \theta(x, 1) \leq \theta([x]) \) is clear. If \( y, z \in [x] \), then \( x \leq y, z \) imply \( (y, 1) = (x \lor y, 1 \lor y) \in \theta(x, 1) \) and \( (z, 1) = (x \lor z, 1 \lor z) \in \theta(x, 1) \), whence \( (y, z) \in \theta(x, 1) \). Thus we get \( \theta([x]) \leq \theta(x, 1) \), by the definition of \( \theta([x]) \).

This proves (2). As any compact congruence \( \theta \) of \( A \) by Corollary 3(3) has the form \( \theta = \theta([s]) \), we get that \( \theta = \theta([s]) \).

Since \( a, b \in [a \land b] \), we obtain \( \theta(a, 1) \lor \theta(b, 1) \leq \theta([a \land b]) = \theta(a \land b, 1) \).

Because \( (a \land b, 1) = (a, 1) \land (b, 1), (a, 1), (b, 1) \in \theta(a, 1) \lor \theta(b, 1) \) imply \( (a \land b, 1) \in \theta(a, 1) \lor \theta(b, 1) \), whence we get also \( \theta(a \land b, 1) \leq \theta(a, 1) \lor \theta(b, 1) \).

Thus (3) holds true. \( \square \)

In view of Lemma 4, if a lattice based variety \( V \) with a constant 1 is 1-regular, then the compact congruences in every free algebra \( F_V([x]) \in V \) are generated by the principal filters \( [t([x])] \) corresponding to a single term \( t([x]) \).

Hence, denoting the congruence induced by the filter \( [t] \) by \( \theta([t]) \), we obtain

Corollary 5. Let \( V \) be a variety of algebras of the form \( A = (L, \land, \lor, 0, 1, F) \) which is 1-regular. Then a finitely presented algebra of \( V \) can be represented (up to isomorphism) by a quotient \( F_V([x]) / \theta([t]) \), where \( t = 1 \) corresponds to equations in the presentation \( S \) that define the congruence \( \sim \) in \( F_V([x]) \).

3 Closure of Finitely Presented Algebras Under Direct Products

In this Section we will consider 1-regular varieties \( V \) of \( \ell \)-algebras of the form \( A = (L, \land, \lor, 0, 1, F) \) satisfying the following properties:

(A) Each \( (L, \land, \lor, 0, 1) \) is a bounded lattice where every complemented element \( a \in L \) is central.

(B) Each algebra \( A \) has a unary term \( g \) such that for any \( v \in L \)

\[
v \land g(v) = 0 \text{ and } g(0) = 1,
\]

and we will prove that the direct product of two finitely presented algebras from this variety \( V \) is also a finitely presented algebra. Observe, that condition

\[
\theta([x]) = \theta(x, 1),
\]

(2)

\[
\theta(a, 1) \lor \theta(b, 1) = \theta(a \land b, 1).
\]

(3)
(A) is related only with the underlying lattices of these algebras and it is satisfied obviously if these lattices are distributive ones.

We will need the following known facts in our proofs. Let $\varphi \in \text{Con}A$. Then to any congruence $\theta \supseteq \varphi$ corresponds a factor congruence

$$\theta/\varphi = \{([a]_\varphi, [b]_\varphi) \in (A/\varphi)^2 \mid (a, b) \in \theta\}$$

of the factor algebra $A/\varphi$; moreover, the mapping

$$\alpha: [\varphi, \nabla_A] \rightarrow \text{Con}A/\varphi, \text{ where } \alpha(\theta) = \theta/\varphi$$

is a lattice isomorphism. Clearly, $\alpha$ maps $\varphi$ into $\triangle_\varphi = \varphi/\varphi$, the 0-element of $\text{Con}A/\varphi$. It is easy to check that for any $a, b \in A$ with $\varphi \leq \theta(a, b)$, $\theta(a, b)/\varphi = \theta([a]_\varphi, [b]_\varphi)$ in $\text{Con}A/\varphi$. First, we prove the following:

**Lemma 6.** If $\mathcal{V}$ is a 1-regular variety of $\ell$-algebras satisfying condition (A), then for any $a, b \in L$ with $a \land b = 0$ we have

$$\theta(a, 1) \land \theta(b, 1) = \theta(a \lor b, 1),$$

and the congruences $\theta(a, 1)/\theta(a \lor b, 1)$ and $\theta(b, 1)/\theta(a \lor b, 1)$ form a factor congruence pair of the factor algebra $A/\theta(a \lor b, 1)$.

**Proof.** Let $\psi = \theta(a \lor b, 1)$. Since $a \lor b \in [a] \cap [b]$, and $\theta(a, 1) = \theta(a)$, $\theta(b, 1) = \theta(b)$ by (2), we get $(a \lor b, 1) \in \theta(a, 1) \land \theta(b, 1)$, i.e. $\psi \leq \theta(a, 1) \land \theta(b, 1)$. Then $\psi = \nabla_A$ implies $\theta(a, 1) \land \theta(b, 1) = \nabla_A$, and we are done. Thus we may assume $\psi \neq \nabla_A$. Now, consider the factor lattice $L/\psi$. We obtain:

$$[a]_\psi \land [b]_\psi = [0]_\psi \text{ and } [a]_\psi \lor [b]_\psi = [a \lor b]_\psi = [1]_\psi.$$

Hence $[a]_\psi$ and $[b]_\psi$ are the complements of each other in $L/\psi$, therefore, they are central elements in $L/\psi$ and define a factor congruence pair

$$\varrho_1 = \{(x)_\psi, (y)_\psi) \in (L/\psi)^2 \mid [x]_\psi \land [a]_\psi = [y]_\psi \land [a]_\psi\},$$

$$\varrho_2 = \{(x)_\psi, (y)_\psi) \in (L/\psi)^2 \mid [x]_\psi \land [b]_\psi = [y]_\psi \land [b]_\psi\}.$$

of $L/\psi$. Since $\mathcal{A}/\psi = (L/\psi, \land, \lor, [0]_\psi, [1]_\psi, F) \in \mathcal{V}$ is an $\ell$-algebra, $\varrho_1, \varrho_2$ are also factor congruences of the algebra $\mathcal{A}/\psi$. As $\ker(\varrho_1) = [[a]_\psi]$ and $\mathcal{A}/\psi$ is 1-regular, by using (2) we get $\varrho_1 = \theta([[a]_\psi]) = \theta([a]_\psi, [1]_\psi) = \theta(a, 1)/\psi$. Similarly, using $\ker(\varrho_2) = [[b]_\psi]$ we get $\varrho_2 = \theta([b]_\psi, [1]_\psi) = \theta(b, 1)/\psi$. Then
we obtain prove that there is a compact congruence $t$ and $F$

Claim 8. We can present the two free algebras, given at the beginning, as follows:

Hence $\varphi$ where $\theta((a, 1) \wedge b, 1) = \varphi(a, 1) \wedge b, 1)$ and $\varphi_2 = \varphi(b, 1) \wedge \theta(a \vee b, 1)$ form a factor congruence pair of the algebra $A/\varphi = A/\theta(a \vee b, 1)$. □

Theorem 7. Let $\mathcal{V}$ be a $1$-regular variety of $\ell$-algebras satisfying the conditions (A) and (B). If $A, B \in \mathcal{V}$ are finitely presented algebras, then their direct product $A \times B$ is also finitely presented.

Proof. By definition, $A$ and $B$ can be represented in the form $\mathcal{F}(m)/\theta_1$ and $\mathcal{F}(n)/\theta_2$, where $\theta_1, \theta_2$ are two compact congruences on free algebras $\mathcal{F}(m) = \mathcal{F}(x) \in \mathcal{V}$ and $\mathcal{F}(n) = \mathcal{F}(y) \in \mathcal{V}$, respectively. We are going to prove that there is a compact congruence $\varphi$ on a free algebra $\mathcal{F}(m+n+1) = \mathcal{F}(x, y, v)$ such that:

$$\mathcal{F}(m)/\theta_1 \times \mathcal{F}(n)/\theta_2 \cong \mathcal{F}(m+n+1)/\varphi$$

We can present the two free algebras, given at the beginning, as follows:

$\mathcal{F}(m)/\theta_1 \cong \mathcal{F}(m+n)/\varphi_1$ and $\mathcal{F}(n)/\theta_2 \cong \mathcal{F}(m+n)/\varphi_2$.

where $\varphi_1, \varphi_2 \in \text{Con}\mathcal{F}(m+n)$. Clearly, $\varphi_1 = \theta_1 \vee \theta((y_1, 1), \ldots, (y_n, 1))$ and $\varphi_2 = \theta_2 \vee \theta((x_1, 1), \ldots, (x_m, 1))$. Since $\theta_1$ and $\theta_2$ are compact congruences, by Corollary 3(3) they have the form $\theta_1 = \theta(s_1, 1)$ and $\theta_2 = \theta(s_2, 1)$, where $s_1 \in \mathcal{F}(x)$ and $s_2 \in \mathcal{F}(y)$. By using Lemma 4 we obtain:

$\varphi_1 = \theta(s_1 \wedge y_1 \wedge \ldots \wedge y_n, 1)$ and $\varphi_2 = \theta(s_2 \wedge x_1 \wedge \ldots \wedge x_m, 1)$.

Hence $\varphi_1$ and $\varphi_2$ are compact congruences. Let $t_1(x, y) = s_1 \wedge y_1 \wedge \ldots \wedge y_n$ and $t_2(x, y) = s_2 \wedge x_1 \wedge \ldots \wedge x_m$. Then $t_1(x, y), t_2(x, y) \in \mathcal{F}(m+n)$, and by (2) we get $\varphi_1 = \theta(t_1(x, y))$ and $\varphi_2 = \theta(t_2(x, y))$. By using (2) and (3) we obtain $\theta(t_1(x, y) \wedge v) = \theta(t_1(x, y), 1 \vee \theta(v, 1) = \theta(v, 1) \vee \theta(t_1(x, y))$. Now let $g$ be the unary term from condition (B). Using (2) and (3) we get:

$\theta(t_2(x, y) \wedge g(v)) = \theta(t_2(x, y), 1 \vee \theta(g(v), 1) = \theta(g(v), 1) \vee \theta(t_2(x, y))$.

As in view of (B), $v \vee g(v) = 0$, we get $(0, v) = (v \wedge g(v), v \wedge 1) \in \theta(g(v), 1))$.

Therefore, one can deduce:

Claim 8.

(i) $\mathcal{F}(m+n+1)/\theta(t_1(x, y) \wedge v) \cong \mathcal{F}(m+n)/\theta(t_1(x, y))$

and

(ii) $\mathcal{F}(m+n+1)/\theta(t_2(x, y) \wedge g(v)) \cong \mathcal{F}(m+n)/\theta(t_2(x, y))$.  

10
As a consequence of Claim 8, we deduce

\[ \mathcal{F}(m + n + 1)/\theta(t_1(x, y) \land v) \cong \mathcal{F}(m + n)/\varphi_1 \cong \mathcal{F}(n)/\theta_1; \]  

(6)

\[ \mathcal{F}(m + n + 1)/\theta(t_2(x, y) \land g(v)) \cong \mathcal{F}(m + n)/\varphi_2 \cong \mathcal{F}(n)/\theta_2. \]  

(7)

Let us consider now the term \( t = (t_1(x, y) \land v) \lor (t_2(x, y) \land g(v)) \) and the principal congruence \( \psi = \theta(t, 1) = \theta[t] \) of the free algebra \( \mathcal{F}(m + n + 1) \).

Now, we are going to prove

\[ \mathcal{F}(m)/\theta_1 \times \mathcal{F}(n)/\theta_2 \cong \mathcal{F}(m + n + 1)/\psi. \]  

(*)

Indeed, let us consider the elements \( t_1(x, y) \land v \) and \( t_2(x, y) \land g(v) \) in \( \mathcal{F}(m + n + 1) \in \mathcal{V} \). By using condition (B), we get \( (t_1(x, y) \land v) \lor (t_2(x, y) \land g(v)) = t_1(x, y) \land t_2(x, y) \land (v \lor g(v)) = 0 \). Then the assumption and Lemma 6 imply that

\[ \theta((t_1(x, y) \land v), 1) \land \theta(t_2(x, y) \land g(v), 1) = \theta(1, v) = \psi, \]

and that \( \mathcal{F}(m + n + 1)/\psi \) is isomorphic to

\[ (\mathcal{F}(m + n + 1)/\psi)/(\theta((t_1(x, y) \land v), 1) \lor (\theta(t_2(x, y) \land g(v), 1)) \cong (\mathcal{F}(m + n + 1)/\psi)/(\theta(t_2(x, y) \land g(v), 1)/\psi). \]

In view of the second isomorphism theorem of algebras we get

\[ (\mathcal{F}(m + n + 1)/\psi)/(\theta((t_1(x, y) \land v), 1) \lor (\theta(t_2(x, y) \land g(v), 1)) \cong \mathcal{F}(m + n + 1)/\theta(t_2(x, y) \land g(v), 1). \]

Since \( \mathcal{F}(m + n + 1)/\theta((t_1(x, y) \land v), 1) = \mathcal{F}(m + n + 1)/\theta(t_1(x, y) \land v) \cong \mathcal{F}(m)/\theta_1 \) and \( \mathcal{F}(m + n + 1)/\theta(t_2(x, y) \land g(v), 1) = \mathcal{F}(m + n + 1)/\theta(t_2(x, y) \land g(v)) \cong \mathcal{F}(n)/\theta_2 \) according to (6) and (7), we obtain \( \mathcal{F}(m + n + 1)/\psi \cong \mathcal{F}(m)/\theta_1 \times \mathcal{F}(n)/\theta_2 \), thus (*) holds.

Hence \( \mathcal{F}(m + n + 1)/\psi \cong A \times B. \) Since the congruence \( \psi \) is defined by a single term \( t \in \mathcal{F}(m + n + 1) \), the algebra \( \mathcal{F}(m + n + 1)/\psi \) is finitely presented, and this completes the proof of our theorem. \( \square \)

4 CLOSURE OF FINITELY PRESENTED PROJECTIVE ALGEBRAS UNDER DIRECT PRODUCTS

Let \( \mathcal{V} \) be a variety of \( \ell \)-algebras, and consider the following condition:

(C) Each algebra \( A = (L, \land, \lor, 0, 1, F) \in \mathcal{V} \) has two unary terms \( h \) and \( h' \), such that for every \( v \in L \), \( h(v) \) and \( h'(v) \) are the complements of each other in the lattice \( L \), and \( h(0) = h'(1) = 1. \)
Clearly from condition (C) it follows also:
\[ h(1) = h'(0) = 0. \]
Our aim is to prove the following

**Theorem 9.** Let \( \mathcal{V} \) be a 1-regular variety of \( \ell \)-algebras satisfying the conditions (A), (B) and (C). If \( A, B \in \mathcal{V} \) are finitely presented projective algebras, then \( A \times B \) is also a finitely presented projective algebra of \( \mathcal{V} \).

In order to simplify our proof we introduce the notations:
\[ x^- = x \wedge h'(v), x^+ = x \wedge h(v) \]
and
\[ a \cdot b = a^- \lor b^+ = (a \wedge h'(v)) \lor (b \wedge h(v)) \]
for arbitrary \( a, b, v, x \in L \). Then clearly, \( x^- = x^- \) and \( x^+ = x^+ \).

First, we will prove

**Lemma 10.** Let \( \mathcal{V} \) be a variety of \( \ell \)-algebras satisfying the conditions (A) and (C). Then for any \( k \)-ary polynomial \( f \) the identity
\[ f(x_1, \ldots, x_k) \cdot f(y_1, \ldots, y_k) = f(x_1 \cdot y_1, \ldots, x_k \cdot y_k) \]
is satisfied in \( \mathcal{V} \).

**Proof.** The assumptions (A) and (C) imply that in any algebra \( (L, \wedge, \lor, 0, 1, F) \) from \( \mathcal{V} \) the elements \( h(v) \) and \( h'(v) \) form a central pair of the underlying lattice \( L \), and hence they define a factor congruence pair \( \equiv_{(1)}, \equiv_{(2)} \) of \( L \) as follows:
\[ x \equiv_{(1)} y \iff x \wedge h(v) = y \wedge h(v) \iff x^+ = y^+, \]
\[ x \equiv_{(2)} y \iff x \wedge h'(v) = y \wedge h'(v) \iff x^- = y^- . \]
Since \( A = (L, \wedge, \lor, 0, 1, F) \) is an \( \ell \)-algebra, by definition any factor congruence of \( L \) is preserved by any polynomial of \( A \), therefore, from \( x_1 \equiv_{(1)} x_1^+, \ldots, x_k \equiv_{(1)} x_k^+ \) it follows \( f(x_1, \ldots, x_k) \equiv_{(1)} f(x_1^+, \ldots, x_k^+) \), that is,
\[ f(x_1, \ldots, x_k)^+ = f(x_1^+, \ldots, x_k^+)^+ . \] (8)
Similarly we prove:
\[ f(x_1, \ldots, x_k)^- = f(x_1^-, \ldots, x_k^-)^- . \] (9)

Because \( h(v) \) and \( h'(v) \) form a central pair of \( L \), for any \( a, b \in L \) we obtain:
\[ (a \wedge h'(v)) \lor (b \wedge h(v)) \equiv_{(1)} b \wedge h(v) \] and \( (a \wedge h'(v)) \lor (b \wedge h(v)) \equiv_{(2)} a \wedge h'(v) . \)

\* We wish to thank the referee for suggesting the simplification in the proof that follow.
In other words, we get:

\[ a \ast b \equiv_{(1)} b^+ \quad \text{and} \quad a \ast b \equiv_{(2)} a^- . \]  

(10)

These relations imply:

\[ (a \ast b)^+ = b^+ = b^+ \quad \text{and} \quad (a \ast b)^- = a^- = a^- . \]  

(11)

Hence in view of (11) we have

\[
(f(x_1, ..., x_k) \ast f(y_1, ..., y_k))^+ = (f(y_1, ..., y_k))^+ ,
\]

(12)

\[
(f(x_1, ..., x_k) \ast f(y_1, ..., y_k))^- = (f(x_1, ..., x_k))^-. \]  

(13)

(10) yields also that the relations

\[ x_i \ast y_i \equiv_{(1)} y_i^+ \quad \text{and} \quad x_i \ast y_i \equiv_{(2)} x_i^- \quad \text{hold for all} \quad i \in \{1, ..., k\}. \]

Since the factor congruences \( \equiv_{(1)} \) and \( \equiv_{(2)} \) are preserved by \( f \), we get that

\[
f(x_1 \ast y_1, ..., x_k \ast y_k) \equiv_{(1)} f(y_1^+, ..., y_k^+) , \quad \text{and} \]

\[
f(x_1 \ast y_1, ..., x_k \ast y_k) \equiv_{(2)} f(x_1^-, ..., x_k^-) , \]  

whence by using (8) and (9) we obtain

\[
f(x_1 \ast y_1, ..., x_k \ast y_k)^+ = f(y_1^+, ..., y_k^+) = f(y_1, ..., y_k)^+ , \]  

(14)

\[
f(x_1 \ast y_1, ..., x_k \ast y_k)^- = f(x_1^-, ..., x_k^-) = f(x_1, ..., x_k)^- . \]  

(15)

Now, from (12) – (15) we infer:

\[
f(x_1 \ast y_1, ..., x_k \ast y_k)^+ = (f(x_1, ..., x_k) \ast f(y_1, ..., y_k))^+ , \]  

(16)

\[
f(x_1 \ast y_1, ..., x_k \ast y_k)^- = (f(x_1, ..., x_k) \ast f(y_1, ..., y_k))^- . \]  

(17)

Finally, observe that the maps \( x \mapsto x^- , \quad x \in L \) and \( x \mapsto x^+ , \quad x \in L \) are enough to recover an element \( x \in L \). Indeed, since \( h(v) , h'(v) \) is a central pair in \( L \), for each \( x \in L \) the equality \( x = (x \wedge h'(v)) \lor (x \wedge h(v)) = x^- \lor x^+ \) holds. Hence for any \( a , b \in L \), if \( a^- = b^- \) and \( a^+ = b^+ \), then

\[ a = a^- \lor a^+ = b^- \lor b^+ = b . \]

Therefore, in view of this observation (16) and (17) imply:

\[
f(x_1 \ast y_1, ..., x_k \ast y_k) = f(x_1, ..., x_k) \ast f(y_1, ..., y_k) , \]

and this is exactly the identity that we intended to prove. \( \square \)

\textit{Proof of Theorem 9.} Let \( A , B \in V \) two finitely presented projective algebras. Then, by Theorem 7, \( D = A \times B \) is finitely presented. Therefore, as we
pointed out in Section 2, to prove that $D$ is projective, it suffices to show that it is a retract of a free algebra of $V$. Because $A$ and $B$ are retracts of some finitely generated free algebras $F(m)$ and $F(n)$, respectively, $D$ is a retract of a direct product $F(m) \times F(n)$. Clearly, by showing that $F(m) \times F(n)$ is a retract of a free algebra $F \in V$, it follows that $D$ is also a retract of $F$, and our proof is finished.

Hence, in what follows, we will show that $F(m) \times F(n)$ is a retract of the free algebra $F(m+n+1)$, that is, in case of free algebras $F(m) = F(x)$, $F(n) = F(y)$ and $F(m+n+1) = F(x,y,v)$ there are two homomorphisms $m$ and $q$ such that

$$m: F(m) \times F(n) \to F(m+n+1),$$

$$q: F(m+n+1) \to F(m) \times F(n),$$

and $q \circ m = id$ (on $F(m) \times F(n)$).

In order to prove this, let us define for arbitrary $t_1(x) \in F(x), t_2(y) \in F(y)$:

$$m((t_1(x), t_2(y))) = t_1(x) \ast t_2(y),$$

and consider the mappings

$$\tau_1(x_i) = x_i, 1 \leq i \leq m; \tau_1(y_j) = 1, 1 \leq j \leq n; \tau_1(v) = 1$$

and

$$\tau_2(x_i) = 0, 1 \leq i \leq m; \tau_1(y_j) = y_j, 1 \leq j \leq n; \tau_1(v) = 0.$$

Then there are some (unique) homomorphisms $\sigma_1: F(m+n+1) \to F(m)$ and $\sigma_2: F(m+n+1) \to F(n)$, such that $\sigma_1$ is an extension of $\tau_1$ and $\sigma_2$ is an extension of $\tau_2$, respectively. Now, we define

$$q((t(x, y), v)) := (\sigma_1(t(x, y)), \sigma_2(t(x, y, v))).$$

Clearly, $q: F(x,y,v) \to F(x) \times F(y)$ is a homomorphism, and

$$q((t(x, y, v)) = (t(\tau_1(x), \tau_1(y), \tau_1(v)), t(\tau_2(x), \tau_2(y), \tau_2(v))).$$

According to the definition of $\tau_1$ and $\tau_2$ we can use the shortened notation:

$$q((t(x, y, v)) = (t(x, y/1, v/1), t(x/0, y,v/0)).$$

First, let us prove that $q \circ m$ is equal to the identity map on $F(m) \times F(n)$. Take $(t_1(x), t_2(y)) \in F(x) \times F(y)$ arbitrary. Then $(q \circ m)((t_1(x), t_2(y)) =

= q((t_1(x) \ast t_2(y))) = q((t_1(x) \wedge h'(v)) \cup (t_2(y) \wedge h(v))) =

= ((t_1(x) \wedge h'(1)) \cup (t_2(y/1) \wedge h(1)), (t_1(x/0) \wedge h'(0)) \cup (t_2(y) \wedge h(0))).$

Since $h'(1) = 1$, $h(1) = 0$ and $h'(0) = 0$, $h(0) = 1$, we obtain

$$(q \circ m)((t_1(x), t_2(y)) = (t_1(x), t_2(y)).$$

Now, it remains to show only that $m$ is a homomorphism, that is, any $k$-ary polynomial $f$ of $F(m) \times F(n)$, commutes with $m$. Let $(t_1^{(1)}(x), t_2^{(1)}(y)), \ldots, (t_1^{(k)}(x), t_2^{(k)}(y)) \in F(m) \times F(n)$. Then by definition

$$f((t_1^{(1)}(x), t_2^{(1)}(y)), \ldots, (t_1^{(k)}(x), t_2^{(k)}(y))) =

= f(t_1^{(1)}(x), \ldots, t_1^{(k)}(x)), f(t_2^{(1)}(y), \ldots, t_2^{(k)}(y)),$$

so we have to prove that
\[ m(f(t_1^{(1)}(x), \ldots, t_1^{(k)}(x)), f(t_2^{(1)}(y), \ldots, t_2^{(k)}(y))) = \\
= f(m(t_1^{(1)}(x), t_2^{(1)}(y)), \ldots, m(t_1^{(k)}(x), t_2^{(k)}(y))), \text{ that is,} \\
f(t_1^{(1)}(x), \ldots, t_1^{(k)}(x)) * f(t_2^{(1)}(y), \ldots, t_2^{(k)}(y)) = \\
= f(t_1^{(1)}(x) * t_2^{(1)}(y), \ldots, t_1^{(k)}(x) * t_2^{(k)}(y)). \quad (18) \]

Since \( V \) satisfies conditions (C) and (A), we can apply Lemma 10 with \( x_1 = t_1^{(1)}(x), \ldots, x_k = t_1^{(k)}(x), y_1 = t_2^{(1)}(y), \ldots, y_k = t_2^{(k)}(y) \), and this yields (18). Hence \( m \) is a homomorphism, and this completes our proof. \( \square \)

Since any bounded distributive lattices satisfies condition (A), the following consequence is immediate:

**Corollary 11.** Let \( V \) be a variety of 1-regular distributive \( \ell \)-algebras satisfying conditions (B) and (C). If \( A, B \in V \) are finitely presented projective algebras, then \( A \times B \) is also a finitely presented projective algebra.

Finally, by applying the result of Ghilardi and Sacchetti [14], i.e. Theorem 1, we obtain:

**Theorem 12.** Let \( V \) be a variety of 1-regular \( \ell \)-algebras. If the conditions (A), (B) and (C) are satisfied by \( V \), then unification in \( V \) is filtering.

**An existing application for Heyting algebras.** We observe that the existing result for Heyting algebras, see [6], [7], is a special case of the above Theorem 12. As we pointed out in the preliminaries, Heyting algebras are \( \ell \)-algebras. All Heyting algebras contain the constant 1 and the operation \( \to \) satisfies the condition

\[ x \leq y \iff x \to y = 1, \]

thus they are also 1-regular. Since Heyting algebras are based on bounded distributive lattices, they clearly satisfy condition (A). Hence, to apply our results, it remains to check conditions (B) and (C). Consider the variety \( \mathbb{H} \) of Heyting algebras \( (A, \wedge, \vee, \to, 0, 1) \). Since \( \neg v = v \to 0 \), for all \( v \in A \), \( \mathbb{H} \) clearly satisfies condition (B) with \( g(v) = \neg v \) (a pseudocomplement). Now let \( S \) be a subvariety of \( \mathbb{H} \) which satisfies the Stone identity:

\[ \neg \neg x \vee \neg x = 1. \]

Then condition (C) is satisfied by setting

\[ h(v) = \neg v, h'(v) = \neg \neg v. \]

Hence, the unification in \( S \) is filtering.
Moreover, since \((Th(\neg x \lor \neg x = 1), Th(2^2 \oplus 1))\) is the splitting pair for the lattice of theories of Heyting algebras one gets that: unification in a variety \(\mathcal{V} \subseteq \mathcal{H}\) is filtering iff the Stone identity holds in \(\mathcal{V}\), see [6].

The advantages of the present results are that they are preserved by expansions of algebras with compatible operations (note that any compatible operation is center-preserving), they can be applied to algebras without EDPC and to algebras based on non-distributive lattices, as shown below.

5 FILTERING UNIFICATION IN SOME VARIETIES OF RESIDUATED LATTICES

An algebra \(\mathcal{L} = (L, \land, \lor, \oplus, \to, 0, 1)\) is called an integral bounded commutative residuated lattice, IBCRL or simply bounded residuated lattice, if
1. \((L, \land, \lor, 0, 1)\) is a bounded lattice;
2. \((L, \oplus)\) is a commutative monoid with unit element 1;
3. \(x \oplus y \leq z \iff x \leq y \to z\), for all \(x, y, z\).

It is known that the following distributivity rule holds, see e.g. [11]:
\[(x \lor y) \land z = x \land (y \land z),\]
moreover, we define \(\neg x = x \to 0\). The variety IBCRL is 1-regular, and it was proved in [20] that in bounded residuated lattices, although the underlying lattices need not to be distributive, the complements are unique. In what follows, we are going to prove that any bounded residuated lattice is an \(\ell\)-algebra satisfying condition (A). First, we present some known properties of these algebras.

The results of Kowalski and Ono [20]: Lemma 1.3. and [20]: Lemma 1.6., see also [11], can be summarized as the following Kowalski - Ono Lemma:

**Lemma 13.** Let \((L, \land, \lor, \oplus, \to, 0, 1)\) be a bounded residuated lattice and let \(a \in L\) have a complement \(b \in L\). Then the following hold:
(i) If \(c\) is a complement of \(a\) in \(L\) then \(c = b\);
(ii) \(\neg a = b\) and \(\neg b = a\);
(iii) \(a^2 = a\);
(iv) \(a \land x = a \land x\), for any \(x \in L\).

As an immediate consequence we obtain:

**Corollary 14.** If \(a \in L\) is a complemented element of a residuated lattice \((L, \land, \lor, \oplus, \to, 0, 1)\), then \(a \land (x \lor y) = (a \land x) \lor (a \land y)\), for every \(x, y \in L\).
Proof. Take any \( x, y \in L \). In view of Lemma 13(iv) we obtain:
\[
a \land (x \lor y) = a \lor (x \lor y) = a \lor x \lor a \lor y = (a \land x) \lor (a \land y).
\]

An \( i \)-filter (implicative filter) of an (integral commutative) residuated lattice \( L = (L, \land, \lor, \circ, \rightarrow, 1) \) is a nonempty subset \( F \subseteq L \) which satisfies the following conditions:
(a) \( F \) is an order filter of the lattice \( L \);
(b) If \( x, y \in F \) then \( x \circ y \in F \).

In view of [5] an equivalence \( \theta \subseteq L \times L \) is a congruence of the algebra \( L \) if and only if the \( \theta \)-equivalence class \( \theta[1] \) is an \( i \)-filter, and this is equivalent to
\[
(x, y) \in \theta \iff (x \rightarrow y) \land (y \rightarrow x) \in \theta[1].
\]

Now let \( z, \overline{z} \in L \) be the complements of each other in a bounded lattice \( L \). In view of [21] (see also [23]), \( z, \overline{z} \in \text{Cen}(L) \) if and only if the following hold:
(c) \( x = (x \land z) \lor (x \land \overline{z}) \), for any \( x \in L \);
(d) \( \overline{z} \land (u \circ z) = u \), for any \( u \leq \overline{z} \) and \( z \land (v \lor \overline{z}) = v \), for any \( v \leq z \).

**Theorem 15.** Any bounded residuated lattice is an \( \ell \)-algebra satisfying condition (A).

Proof. Let \( L = (L, \land, \lor, \circ, \rightarrow, 0, 1) \) be a bounded residuated lattice and \( c \in L \) a central element of the lattice \( L \). We are going to prove that the congruence \( \theta_c = \{(x, y) \in L^2 \mid x \land c = y \land c\} \) of \( L \) is also a congruence of the algebra \( L \). First, observe that \( \theta_c[1] = \{x \in L \mid x \land c = c\} = [c] \), that is, \( \theta_c[1] \) is principal filter of \( L \). Now take \( x, y \in \theta_c[1] \). Then \( c \leq x, y \).

Since \( c \) is a complemented element of \( L \), by using Lemma 13(iv) we get \( c = c \land y = c \circ y \leq x \circ y \), and hence \( x \circ y \in [c] = \theta_c[1] \). Thus \( \theta_c[1] \) is an \( i \)-filter in \( L \). Now let \( (x, y) \in \theta_c \). Then \( x \land c = y \land c \) implies \( x \circ c = y \circ c \).

We claim that
\[
(x \rightarrow y) \land (y \rightarrow x) \geq c.
\]

Indeed, \( c \circ x = y \circ c = y \land c \leq y \), yields \( c \leq x \rightarrow y \) and \( c \circ y = c \circ x = c \land x \leq x \) gives \( c \leq y \rightarrow x \). Thus \( (x \rightarrow y) \land (y \rightarrow x) \geq c \).

Conversely, suppose \( (x \rightarrow y) \land (y \rightarrow x) \geq c \). Then \( c \leq x \rightarrow y \) and \( c \leq y \rightarrow x \) imply \( c \land x = c \circ x \leq y \) and \( c \land y = c \circ y \leq x \). Hence \( x \land c \leq y \land c \) and \( y \land c \leq x \land c \), thus we get \( x \land c = y \land c \), i.e. \( (x, y) \in \theta_c \).

As we proved that \( (x, y) \in \theta_c \iff (x \rightarrow y) \land (y \rightarrow x) \in [c] = \theta_c[1] \), this means that \( \theta_c \) is a congruence of the algebra \( L \). Therefore, \( L \) is an \( \ell \)-algebra.
Further, let \( a, \pi \) be the complements of each other in the lattice \( L \). To prove that \( L \) satisfies condition (A), we have to show only that \( a, \pi \in \text{Cen}(L) \). We will show this by verifying that conditions (d) and (c) hold.

Indeed, let \( u \leq \pi \) and \( v \leq a \). Then, in view of Corollary 14, we obtain:

\[
\pi \land (u \lor a) = (\pi \land u) \lor (\pi \land a) = u \quad \text{and} \quad a \land (v \lor \pi) = (a \land v) \lor (a \land \pi) = v,
\]

and hence (d) is satisfied.

Finally, to prove (c) take any \( x \in L \). Since both \( a \) and \( \pi \) are complemented elements, in view Lemma 13(iv), we have \( x \circ a = x \land a \) and \( x \circ \pi = x \land \pi \).

Thus we can write:

\[
x = x \circ 1 = x \circ (a \lor \pi) = x \circ a \lor x \circ \pi = (x \land a) \lor (x \land \pi).
\]

Since this proves (c), in view of [21] we obtain \( a, \pi \in \text{Cen}(L) \), and hence \( L \) satisfies condition (A).

Now let \( L = (L, \land, \lor, \circ, \to, 0, 1) \) be a residuated lattice satisfying

\[
\neg \neg x \land x = 0, \quad \text{for all} \ x \in L. \quad \text{(P)}
\]

Such a residuated lattice is always pseudocomplemented, i.e. for any \( x, y \in L \)

\[
y \land x = 0 \iff y \leq \neg x
\]

see e.g. [5]. Therefore, a bounded residuated lattice is called pseudocomplemented if it satisfies condition (P). Observe, that defining \( g(v) = \neg v \), for all \( v \in L \) condition (B) holds for \( L \) and hence we obtain:

**Corollary 16.** If \( \mathcal{A}, \mathcal{B} \) are finitely presented pseudocomplemented residuated lattices then \( \mathcal{A} \times \mathcal{B} \) is also finitely presented.

A residuated lattice \( L \) (or a variety) is called Stonean if it satisfies the Stone identity

\[
\neg \neg x \lor \neg x = 1, \quad \text{for all} \ x \in L. \quad \text{(S)}
\]

Since \( x \land \neg x \leq \neg \neg x \land \neg x = (\neg x \lor \neg \neg x) = \neg 1 = 0 \), each Stonean residuated lattice is pseudocomplemented and, obviously, bounded. Hence we have:

**Theorem 17.** Let \( \mathcal{V} \) be a variety of Stonean (integral commutative) residuated lattices. Then the unification in \( \mathcal{V} \) is filtering.

**Proof.** By Theorem 15 any algebra \( L \in \mathcal{V} \) is a 1-regular \( \ell \)-algebra which satisfies condition (A). If axiom (S) is satisfied then \( L \) is pseudocomplemented, and hence condition (B) is satisfied with \( g(v) = \neg v \).
Now, let \( h(v) = \neg v \), \( h'(v) = \neg \neg v \). Since \( h(v) \land h'(v) = 0 \) and \( h(v) \lor h'(v) = 1 \), \( h(v) \) and \( h'(v) \) are the complements of each other, and hence condition (C) is also satisfied by \( \mathcal{L} \). Then, in view of Theorem 12, unification in \( \mathcal{V} \) is filtering. \( \square \)

In view of Theorem 9, from the above proof it follows also

**Corollary 18.** Let \( \mathcal{V} \) be a variety of integral commutative residuated lattices which satisfies axiom (S). If \( \mathcal{A}, \mathcal{B} \in \mathcal{V} \) are finitely presented projective algebras, then \( \mathcal{A} \times \mathcal{B} \) is also a finitely presented projective algebra.

**Corollary 19.** Unification in any variety of (integral commutative) residuated lattices satisfying the Stone identity (S) is either unitary or nullary.

Now we will show a partial converse of Theorem 17, that is, some residuated lattices \( \mathcal{A} \) with filtering unification are Stonean. We use the following known properties of residuated lattices:

**Lemma 20.** Let \( \mathcal{V} \) be a variety of integral commutative residuated lattices. Then the following conditions hold in \( \mathcal{V} \).

(i) \( x \to (y \to z) = x \odot y \to z = y \to (x \to z) \) (com),

(ii) \( x \odot y \leq x \land y \), \( x \odot (x \to y) \leq y \), \( \neg x \odot x = 0 \).

(iii) \( x \to (y \land z) = (x \to y) \land (x \to z) \), \( x \to y = 1 \iff x \leq y \),

(iv) if \( x \leq y \), then \( \neg y \leq \neg x \);

(v) \( x \leq \neg \neg x \),

(vi) if \( \neg x \odot \neg x = \neg x \), i.e. \( \neg x \) is idempotent, then the principal filter \( [\neg x] \) is an i-filter and \( \theta([\neg x]) \) is a congruence given by an i-filter.

(vii) if \( \mathcal{A} \) is Stonean, then \( \neg x \odot \neg x = \neg x \) holds in \( \mathcal{A} \), for any \( \mathcal{A} \in \mathcal{V} \).

For (vi), observe that for any \( a, b \in [\neg x] \) we have \( a, b \geq \neg x \) and this implies \( a \odot b \geq \neg x \odot \neg x = \neg x \), hence \( a \odot b \in [\neg x] \).

We show (vii): \( \neg x = \neg x \odot 1 = \neg x \odot (\neg x \lor \neg \neg x) = \neg x \odot \neg x \lor \neg x \odot (\neg x \lor \neg \neg x) = \neg x \odot (\neg x \lor \neg x) = \neg x \odot \neg x \).

We prove a partial converse of Theorem 17.
Theorem 21. Let \( \mathcal{V} \) be a variety of (commutative integral bounded) residuated lattices. Then the following are equivalent:

(i) unification in \( \mathcal{V} \) is filtering and the identity \( \neg x \circ \neg x = \neg x \) holds in the variety \( \mathcal{V} \),

(ii) \( \mathcal{V} \) is Stonean.

Proof. (i) \( \Rightarrow \) (ii) is clear by Theorem 17 and Lemma 20 (vii).

(i) \( \Rightarrow \) (ii). Let \( \mathcal{V} \) be a variety of residuated lattices. All equations written below are related to \( \mathcal{V} \). We use the well known fact that any mapping on variables \( \{x_1, \ldots, x_k\} \) has a unique extension to a homomorphism on the free algebra \( \mathcal{F}(x_1, \ldots, x_k) \) generated by \( \{x_1, \ldots, x_k\} \).

Let us consider a unification problem \( (x \lor \neg x, 1) \), where \( x \) is an arbitrary variable. The problem has two unifiers: \( \lambda_0(x) = 0 \) and \( \lambda_1(x) = 1 \).

Since unification in \( \mathcal{V} \) is filtering there is a unifier \( \mu \) for it: \( \mu(x \lor \neg x) = 1 \), and the unifier \( \mu \) is more general then \( \lambda_0 \) and \( \lambda_1 \). This means that there are substitutions \( \sigma_0, \sigma_1 \) such that for any variable \( z \), \( \sigma_1(z) = 1 \) and \( \sigma_0(z) = 0 \), and hence \( \neg \sigma_0 \mu(z) = 1 \).

Let \( \{x_1, \ldots, x_n\} = x \) be the set of all variables in the term \( \mu(x) \), let \( y \) be a fresh variable and let \( \mathcal{F}(x, y) \) be the free lattice generated by \( x \) and \( y \).

Since, by assumption, the elements \( \neg \neg y \) and \( \neg y \) are idempotent, they generate two i-filters \( \neg \neg y \), \( \neg y \) i.e. two congruences \( \theta([-\neg y]) \) and \( \theta([-y]) \) in \( \mathcal{F}(x, y) \). Later in the proof we will define a substitution \( \tau \) on the set \( \{x_1, \ldots, x_n\} \) such that the following conditions hold:

(D) \( (\tau(x), \sigma_1(x)) \in \theta([-\neg y]) \) and
(E) \( (\tau(x), \sigma_0(x)) \in \theta([-y]) \).

Since the congruences \( \theta([-\neg y]) \) and \( \theta([-y]) \) preserve operations, we get:

(D’) \( \neg \neg y \leq (\tau(\mu(x)) \leftrightarrow \sigma_1(\mu(x))) \) and
(E’) \( \neg y \leq (\neg \tau(\mu(x)) \leftrightarrow \neg \sigma_0(\mu(x))) \).

By the fact that \( \sigma_1(\mu(x)) = 1 \) and \( \neg \sigma_0(\mu(x)) = 1 \), we get

(D’) \( \neg y \leq \tau(\mu(x)) \) and
(E’) \( \neg y \leq \neg \tau(\mu(x)) \). By (20) (iv),(v) we can write (D’) as follows:

(D’’) \( \neg \tau(\mu(x)) \leq \neg \neg y = \neg y \). By (v), it follows from (E’) that

\( 1 = \neg y \rightarrow \neg \tau(\mu(x)) = \tau(\mu(x)) \rightarrow \neg y \), thus

(E’) \( \tau(\mu(x)) \leq \neg y \).

Now, by taking joins on both sides of (D’) and (E’) we get

\( 1 = \tau(\mu(x \lor \neg x)) = \neg \tau(\mu(x)) \lor \tau(\mu(x)) \leq \neg y \lor \neg y \), as required.
Consider the following substitution:

\[ \tau(x_j) := (\neg y \to \sigma_1(x_j)) \land (\neg y \to \sigma_0(x_j)). \]

For simplicity of the notation we set: \( t = \tau(x_j), s_0 = \sigma_0(x_j) \) and \( s_1 = \sigma_1(x_j) \). Moreover, we have \( t = (\neg y \to s_1) \land (\neg y \to s_0) \). It remains only to show that (D) and (E) hold (using Lemma 20). By the definition of the congruences \( \theta([\neg y]) \) and \( \theta([-y]) \) we get the equivalent forms of (D) \( \neg y \leq s_1 \iff t \), and of (E) \( \neg y \leq s_0 \iff t \).

For the proof of (D), using 20 (iii): \( a \leq b \iff 1 = a \to b \), we show that

\[(*) 1 = \neg y \to (s_1 \leftrightarrow t) = [\neg y \to (s_1 \to t)] \land [\neg y \to (t \to s_1)]. \]

Now consider the second 'conjunct' of \((*)\) by (i), (ii) and (iii) of Lemma 20

\[\neg y \to (s_1 \land \neg y \to s_0) = \neg y \to (\neg y \to (s_1 \to s_0)) = \neg y \to (s_1 \land \neg y \to s_0) = 1. \]

Now consider the second 'conjunct' of \((*)\): \( [\neg y \to (t \to s_1)] \); we show it is equal to 1. By the definition of \( t \) we show \( \neg y \to (t \to s_1) = [\neg y \to (s_1 \to t)] \land [\neg y \to (t \to s_1)] = 1. \)

For the proof of (E), similarly we show that:

\[(**) 1 = \neg y \to (s_0 \leftrightarrow t) = [\neg y \to (s_0 \to t)] \land [\neg y \to (t \to s_0)]. \]

Consider the first 'conjunct' of \((***)\): \( [\neg y \to (s_0 \to t)] \). By the definition of \( t \) we have, by (iii), two cases:

1) \( \neg y \to (s_0 \to (\neg y \to s_1)) = \neg y \to (\neg y \to (s_0 \to s_1)) = 1 \),\n
2) \( \neg y \to (s_0 \to (\neg y \to s_0)) = \neg y \to (\neg y \to (s_0 \to s_0)) = 1 \).

Now consider the second 'conjunct' of \((***)\): \( [\neg y \to (t \to s_0)] \). By the definition of \( t \) we obtain

\[ \neg y \to (t \to s_0) = [\neg y \to (s_0 \to t)] \land [\neg y \to (t \to s_0)] = \]

\[ = [(\neg y \to s_1) \land (\neg y \to s_0)] \to (\neg y \to s_0) = 1. \]

Hence (E) is proved. This ends the proof of the theorem. \( \square \)

**Remark.** Without assuming that the identity \( \neg x \circ \neg x = \neg x \) holds in the variety, filtering unification does not imply that the variety is Stonean. Consider a three-element MV-algebra \( \{0,1/2,1\} \). Unification in the variety \( \mathcal{V}_3 \) generated by it is unitary, see [8], hence it is filtering, but \( \mathcal{V}_3 \) is not Stonean and the condition \( \neg x \circ \neg x = \neg x \) does not hold in \( \mathcal{V}_3 \) (put \( x = 1/2 \)).

Note that both unification types 1 and 0 actually occur in some varieties specified in Corollary 18. Heyting algebras satisfying (S) will be called here
Heyting – de Morgan algebras; the variety of Heyting – de Morgan algebras has unitary unification. Some of its subvarieties e.g. all the varieties of Gödel algebras have unitary unification, Ghilardi [13] showed that there are subvarieties of Heyting – de Morgan algebras in which unification is nullary.

M. Kondo [19] studied strict residuated lattices, i.e. such lattices $L$, that

\[ \neg(x \odot y) = \neg x \lor \neg y, \text{ for all } x, y \in L. \]  

(\text{Str})

which generalize well known SMTL-algebras and SBL-algebras, i.e. MTL-algebras and BL-algebras, respectively, satisfying (Str); a t-norm $\odot$ satisfying \text{(Str)} is called \textit{strict}. By [19] Prop.3, a residuated lattice satisfies (Str) iff it is Stonean.

An operation $\neg$ is called a \textit{Gödel negation} if $\neg x = 0$, for $x \neq 0$, and $\neg 0 = 1$.

**FACT 1.** In any residuated lattice $L$ the following conditions are equivalent:

(a) $L$ has no (non-trivial) zero divisors: $x \odot y = 0 \Rightarrow (x = 0 \text{ or } y = 0)$,

(b) negation $\neg x = x \to 0$ in $L$ is Gödel negation.

For (a) $\to$ (b) see [19]. For the converse, assume that 0 has non-trivial divisors, say $z \odot x = 0$ and $z \neq 0 \neq x$; then by $z \odot x = 0 \iff z \leq \neg x$, we have $0 \neq z \leq \neg x$, i.e. $\neg$ is not Gödel negation, a contradiction.

**FACT 2.** For any subdirectly irreducible (s.i.) bounded residuated lattice $L$ the following conditions are equivalent\footnote{In [19] FACT 2 is proved for a chain $L$.}:

(i) $L$ satisfies \text{(Str)}, or $L$ is Stonean,

(ii) $L$ has no zero divisors,

(iii) the negation $\neg x = x \to 0$ in $L$ is a Gödel negation,

We show (i) $\to$ (ii). Suppose that $x \odot y = 0$. Then $1 = \neg(x \odot y) = \neg x \lor \neg y$, but since $L$ is subdirectly irreducible, Prop.1.4 in [20] gives $\neg x = 1$ or $\neg y = 1$, hence $x = 0$ or $y = 0$. (ii) $\to$ (iii) and (iii) $\to$ (i) are similar to Proposition 4. in [19].

**Corollary 22.** Let $\mathcal{V}$ be a variety of (commutative integral bounded) residuated lattices. Then the following are equivalent:

(i) $L$ satisfies \text{(Str)}, i.e. $L$ is Stonean, for every $L \in \mathcal{V}$,

(ii) every subdirectly irreducible (s.i.) member of $\mathcal{V}$ has no zero divisors,

(iii) the negation $\neg x = x \to 0$ in every s.i. $L \in \mathcal{V}$ is the Gödel negation.

Hence lack of zero divisors in s.i. residuated lattices that generate the variety $\mathcal{V}$, determine filtering unification in $\mathcal{V}$. In general, since $x \odot \neg x = 0$, the free algebra in $\mathcal{V}$ has zero divisors, it generates $\mathcal{V}$, but it is not s.i.
Corollary 23. Let $\mathcal{V}$ be a variety of (commutative integral bounded) residuated lattices. Then unification in $\mathcal{V}$ is filtering and the identity $\neg x \odot \neg x = \neg x$ holds in $\mathcal{V}$ iff the subdirectly irreducible members of $\mathcal{V}$ have no zero divisors.

Residuated lattices that satisfy prelinearity $(x \rightarrow y) \vee (y \rightarrow x) = 1$ are called MTL-algebras. They form an algebraic semantics for the monoidal t-norm based logic, MTL, the fuzzy logics of all left-continuous t-norms, see [9]. BL-algebras are MTL-algebras satisfying the axiom $(x \wedge y) = x \odot (x \rightarrow y)$. They form an algebraic semantics for P. Hajek’s Basic Fuzzy Logic BL. ΠMTL-algebras (II-algebras) are MTL-algebras (BL-algebras) that satisfy $\neg x \vee ((x \rightarrow x \odot y) \rightarrow y) = 1$. BL-algebras with $\odot = \wedge$ are called Gödel algebras. SMTL-algebras (SBL-algebras) additionally satisfy (Str). For various varieties of residuated lattices see e.g. [11], [20].

It is known, see e.g. [5], 1.9, 1.8, that MTL-algebras are subdirect products of bounded residuated chains, and a bounded residuated chain is Stonean if and only if it is pseudocomplemented. Thus we obtain

Corollary 24. Unification is either unitary or nullary in the following varieties: all varieties of SMTL-algebras, MTL-algebras with Gödel negation, SBL-algebras, ΠMTL-algebras, etc.

Many examples of such varieties can be found in [16] and [17].

Example A. Let $\mathcal{L}_1$ be a residuated lattice which is a five-element Heyting algebra, that is the square $2 \times 2$ with an additional top 1, i.e. $0 < a, b < v < 1$, $a, b$ incomparable, $\neg a = b$, $\neg b = a$ and let $\mathcal{V}_1 \subseteq \mathcal{H}$ be a variety of Heyting algebras containing $\mathcal{L}_1$. Since in Heyting algebras $\odot = \wedge$, we have $\neg x \odot \neg x = \neg x$, but (S) fails in $\mathcal{L}_1$ (and in $\mathcal{V}_1$): $\neg a \vee \neg a = v \neq 1$, $\mathcal{L}_1$ has zero divisors: $a \odot b = 0$. $\mathcal{L}_1$ is the particular splitting of the lattice of subvarieties of $\mathcal{H}$, hence unification in $\mathcal{V}_1$ is not filtering, see [6], [7].

Example B. Let $\mathcal{L}_2$ be a residuated lattice of seven-elements $\|$ which consists of the non-distributive diamond $M_3$ (with the bottom $u$), plus additional top 1 and bottom 0 added, and $x \odot 1 = x$, $x \odot 0 = 0$; $x \odot y = u$ otherwise; i.e. $\mathcal{L}_2$ has no zero divisors. By FACT 2 $\mathcal{L}_2$ is Stonean, hence $\text{HSP}(\mathcal{L}_2)$ is a Stonean variety, thus unification in $\text{HSP}(\mathcal{L}_2)$ is filtering (unitary or nullary). Distributivity fails in $\mathcal{L}_2$.

Example C. Let $\mathcal{L}_3$ be a residuated lattice of seven-elements which consists of the non-modular pentagon $N_5$ (with the bottom $u$), plus additional top 1

\footnote{This is Example 2 of [10].}
and bottom 0 added, and $x \odot 1 = x$, $x \odot 0 = 0$; $x \odot y = u$ otherwise; i.e. $\mathcal{L}_3$ has no zero divisors. By FACT 2 $\mathcal{L}_3$ is Stonean, hence unification in the variety $HSP(\mathcal{L}_3)$ is filtering (unitary or nullary), however modularity fails in $\mathcal{L}_3 \neq$.

6 CONCLUSIONS

We proved that unification is filtering, i.e. unitary or nullary, in any variety of $\ell$-algebras, which satisfy some algebraic conditions (A), (B) and (C). This is true even if the underlaying lattices are not modular. Here $\ell$-algebras are bounded lattice-based algebras in which every term preserves congruences induced by central elements in the underlying lattice. Condition (A) concerns central elements of the underlying lattice, (B) demands a kind of weak pseudocomplement and (C) requires generation of pairs of complemented elements.

This general result is then applied to a particular case of bounded, commutative, integral residuated lattices. It is shown that unification is filtering in Stonean residuated lattices (or equivalently, in varieties generated by s.i. residuated lattices without zero divisors). These conditions are sufficient, but in particular subvarieties they are also necessary. The result on filtering unification apply also to a wide range of fuzzy logics, especially to fuzzy logics in which negation is represented by the pseudocomplement, in particular to logics with Gödel negation.

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REFERENCES


# This example was suggested to us by Peter Jipsen in a private communication.