Logical analogies between intuitionistic fuzzy sets and rough sets

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Abstract. In this note we prove that in the framework of intuitionistic fuzzy sets can be defined connective systems which satisfy some logical rules generalizing the rules of the constructive logic with strong negation (CLSN). As rough set systems defined by a quasiorder serve as models for CLSN, similarly, intuitionistic sets can be viewed as models of its mentioned generalization.

Keywords: Intuitionistic fuzzy set, rough set, Nelson algebra, constructive logic with strong negation, residuated lattice

1 An introduction to intuitionistic fuzzy sets

Intuitionistic sets were introduced by K. T. Attanasov [1] as an extension of fuzzy sets. However, it was observed already by Attanasov [2] that their logical properties differ in some aspects from those of fuzzy sets. In this note we prove that in the framework of intuitionistic fuzzy sets can be defined connective systems which satisfy logical rules analogous to the rules of the constructive logic with strong negation. Since the rough sets are models of the mentioned logic, we will compare their algebraic properties with those of intuitionistic sets.

The intuitionistic fuzzy sets are introduced as pairs $A = (\mu_T, \mu_F)$ of membership functions $\mu_T, \mu_F : U \rightarrow [0, 1]$ defined on a fixed nonempty universe $U$ such that $\mu_T(x) + \mu_F(x) \leq 1$, for all $x \in U$. Therefore, to any element $x \in U$ corresponds a logical value $(\mu_T(x), \mu_F(x))$, where $\mu_T(x)$ expresses the membership value of the element $x$ in the set $A$ and $\mu_F(x)$ the degree of the non-membership of $x$ with respect to $A$. Of course, in this interpretation the value $1 - \mu_T(x) - \mu_F(x)$ denotes a measure of non-determinacy or of the incompleteness of our information. Thus the possible logical values form a lattice

$$L = \{(x_1, x_2) \in [0, 1]^2 \mid x_1 + x_2 \leq 1\}$$

where the partial order is defined as follows:

$$(x_1, x_2) \leq (y_1, y_2) \iff x_1 \leq y_1 \text{ and } x_2 \geq y_2.$$  \quad (P)

Remark 1. It is easy to see that the lattice operations in $L$ have the form
\[(x_1, x_2) \lor (y_1, y_2) = (\max(x_1, y_1), \min(x_2, y_2)), \]
\[(x_1, x_2) \land (y_1, y_2) = (\min(x_1, y_1), \max(x_2, y_2)).\]

Clearly, the least element of \(L\) is \(0_L = (0, 1)\) and its greatest element is \(1_L = (1, 0)\). In Cornelis, Deschrijver and Kerre [4] is proved that \(L\) is a complete sublattice of \([0, 1] \times [0, 1]^d\), and this yields that \(L\) is completely distributive.

**Example 1.** A special area of knowledge management focuses on subjective ontology (epistemology), see Goertz and Mahoney [6]. A key property of subjective ontology is that the ontology is created with a learning process where some information may be incomplete or partial. Thus, the principle of excluded middle is not met in this domain. The subjective ontology is a conceptual model generated by cognitive processes. During the learning process, the aggregated input from the environment is used to build up the conceptual model.

Having an ontology fragment as shown in Fig. 1., every edge correspond to a statement, elementary proposition. The notation is based on the RDF model. A complex arc denotes the specialization (is a) relationship, while the simple arcs refer to the properties of the concepts. As also the properties are concepts, the specialization can be defined among the properties too.

![Sample ontology fragment](image-url)

**Fig. 1.** Sample ontology fragment

It is assumed that at person level a proposition may be true, false or unknown. This approach is very common in data and information systems. For example, the SQL standard in Grant [7] uses a NULL value to denote the unknown value at a field level. For example, the truth value of a statement that a person eats apple can be either T, F or U.

\[
P(eat, Mary, apple) = T
\]
\[
P(eat, Tom, apple) = F
\]
\[
P(eat, Peter, apple) = U
\]
If the engine aggregates the truth values of the different agents to construct the population level truth value, we get a membership like value.

\[
\begin{align*}
m(\text{eat, my friend, fruit, T}) &= .500 \\
m(\text{eat, my friend, fruit, F}) &= .167 \\
m(\text{eat, my friend, fruit, U}) &= .333
\end{align*}
\]

On this way, we get use intuitionistic fuzzy representation of the relationship element \(s\) in the fuzzy ontology. This representation form allow a more efficient decision making in the ontology framework.

**Application areas.** A relevant application area is the fuzzy clustering of a product palette. The goal of the clustering is to determine the groups of similar objects. The generated groups can be used to build up the ontology of the products (see e.g. Liu [12]). This ontology enables an easier management of product categories and it helps us to discover hidden dependencies between the products. The similarity evaluation of the products can be based here on a survey or on a sentiment analysis of natural texts. We can assume that in the survey, the customers can leave some questions blank if they have no information on a given aspect of the products. In both cases, the missing data may occur frequently, this fact justifies the application of an intuitionistic fuzzy model. The intuitionistic model provides additional information to distinguish the missing value from the other not-missing values. Hence the clustering based on this model can provide a more sophisticated partitioning than the base evaluation methods.

Now let \(\mathcal{F}(U)\) stand for the set of all membership functions \(\mu : U \rightarrow [0, 1]\). Defining the operations \(\vee, \wedge\) for any \(f, g \in \mathcal{F}(U)\) as usually (see [13]):

\[
(\mathit{f \vee g})(x) := \max\{f(x), g(x)\} \quad \text{and} \quad (\mathit{f \wedge g})(x) := \min\{f(x), g(x)\},
\]

for all \(x \in U\), we obtain a completely distributive lattice \((\mathcal{F}(U), \vee, \wedge)\). The least element of \(\mathcal{F}(U)\) is the constant \(0\) map on \(U\), denoted by \(0\), and its greatest element is \(1\), the constant \(1\) map. Next, consider the set of intuitionistic fuzzy sets

\[
\mathcal{I}(U) = \{(\mu_T, \mu_F) \in \mathcal{F}(U)^2 | \mu_T(x) + \mu_F(x) \leq 1, \text{for all } x \in U\},
\]

ordered as follows:

\[
(\mu_T, \mu_F) \leq (\nu_T, \nu_F) \iff \mu_T(x) \leq \nu_T(x) \quad \text{and} \quad \nu_F(x) \geq \mu_F(x), \text{for all } x \in U.
\]

\((\mathcal{I}(U), \leq)\) is a complete distributive lattice with least element \((0, 1)\) and greatest element \((1, 0)\) (see e.g. Novák, Perfilieva, & Mockor [13]). We define a unary operation \(\sim\) on \(\mathcal{L}\) and \(\mathcal{I}(U)\) by setting

\[
\sim (x_1, x_2) = (x_2, x_1), \text{ for all } (x_1, x_2) \in \mathcal{L}, \quad \text{and} \quad \sim (\mu_T, \mu_F) = (\mu_F, \mu_S), \text{ for all } (\mu_T, \mu_F) \in \mathcal{I}(U).
\]

In Cornelis, Deschrijver and Kerre [4] \(\sim\) is called the standard negator on \(\mathcal{L}\). Clearly, we have

\[
\sim \sim (x_1, x_2) = (y_1, y_2), \text{ and} \quad (x_1, x_2) \leq (y_1, y_2) \iff \sim (x_1, x_2) \geq \sim (y_1, y_2).
\]
for any \((x_1, x_2), (y_1, y_2) \in \mathcal{L}\). It is obvious that any \((\mu_T, \mu_F), (\nu_T, \nu_F) \in \mathcal{I}(U)\) satisfy the same rules. These properties mean that \((\mathcal{L}, \lor, \land, \sim, (0, 1), (1, 0))\) and \((\mathcal{I}(U), \lor, \land, \sim, (0, 1), (1, 0))\) are De Morgan algebras.

A De Morgan algebra \(\mathcal{A} = (A, \lor, \land, \sim, 0, 1)\) is an algebra such that \((A, \lor, \land)\) is a bounded distributive lattice with a least element 0 and a greatest element 1, and \(\sim\) is a unary operation that satisfies, for all \(x, y \in A\),

\[
\sim(\sim x) = x \quad \text{and} \quad x \leq y \iff \sim x \geq \sim y.
\]

This definition yields that \(\sim\) is an isomorphism between the lattice \(A\) and its dual \(A^d\). Therefore, \(\sim\) satisfies the so-called De Morgan equations:

\[
\sim(x \lor y) = \sim x \land \sim y, \quad \sim(x \land y) = \sim x \lor \sim y. \quad \text{(M)}
\]

Intuitionistic sets can be viewed as a common generalization of the fuzzy sets and of the rough sets. Indeed, fuzzy sets on \(U\) can be interpreted as exact intuitionistic sets \((\mu_T, \mu_F)\) with \(\mu_F = 1 - \mu_T\), i.e., they correspond to the family

\[
\text{Ex}(U) = \{(\mu, 1 - \mu) \mid \mu \in \mathcal{F}(U)\}.
\]

Conversely, any intuitionistic fuzzy set \((\mu_T, \mu_F)\) can be viewed as a pair of "exact" fuzzy sets \((\mu_T, 1 - \mu_T)\) and \((1 - \mu_F, \mu_F)\), and obviously

\[
(\mu_T, 1 - \mu_T) \leq (\mu_T, \mu_F) \leq (1 - \mu_F, \mu_F).
\]

Thus the fuzzy set \((\mu_T, 1 - \mu_T)\) can be considered as a "lower approximation", and \((1 - \mu_F, \mu_F)\) as an "upper approximation" of the intuitionistic set \((\mu_T, \mu_F)\). Hence, we can define an upper approximation operator \(\overline{A}\) and a lower approximation operator \(\underline{A}\) on the set \(\mathcal{I}(U)\) as follows (see Attanasov [2]):

\[
\overline{A} : \mathcal{I}(U) \longrightarrow \text{Ex}(U), \quad \overline{A}(\mu_T, \mu_F) := (1 - \mu_F, \mu_F),
\]

\[
\underline{A} : \mathcal{I}(U) \longrightarrow \text{Ex}(U), \quad \underline{A}(\mu_T, \mu_F) := (\mu_T, 1 - \mu_T)
\]

### 2 T-norms and t-conorms defined on the lattice \(\mathcal{L}\)

In the literature, the lattice \(\mathcal{L} = \{(x_1, x_2) \in [0, 1]^2 \mid x_1 + x_2 \leq 1\}\) of logical values is usually equipped with some additional logical connectives different from \(\lor\) and \(\land\). For this we need to recall the notion of a t-norm and t-conorm defined on a lattice.

A t-norm on a bounded lattice \(L\) (with least element 0\(_L\) and greatest element 1\(_L\)) is an order-preserving, commutative, associative, binary operation \(T : L^2 \rightarrow L\) satisfying \(T(1_L, z) = z\), for all \(z \in L\). Dually, a t-conorm \(S(x, y)\) on \(L\) is an order-preserving, commutative, associative binary operation \(S : L^2 \rightarrow L\), having \(0_L\) as a neutral element. Following the terminology in Cornelis, Deschrijver and Kerre [4], we say that a t-norm \(T\) (respectively a t-conorm \(S\)) on \(\mathcal{L}\) is t-representable if there exists a t-norm \(T\) and a t-conorm \(S\) on the lattice \(([0, 1], \leq)\) such that, for any \(x = (x_1, x_2) \in \mathcal{L}\) and any \(y = (y_1, y_2) \in \mathcal{L}\) we have

\[
T(x, y) = (T(x_1, y_1), S(x_2, y_2)) \quad \text{(2)}
\]
$S(x,y) = (S(x_1,y_1), T(x_2,y_2)).$ \hfill (2')

In [4; Theorem 2] is also proved that for any t-norm $T$ and any t-conorm $S$ on $[0,1]$ satisfying
\begin{equation}
T(a,b) \leq 1 - S(1-a,1-b), \text{ for all } a,b \in [0,1], \tag{3}
\end{equation}
the mappings $T$ and $S$ defined by the formulas (2) and (2') are a t-norm and a t-conorm on $L$, respectively. If the relation in (3) is satisfied with equality, i.e.
\begin{equation}
T(a,b) = 1 - S(1-a,1-b), \text{ for all } a,b \in [0,1], \tag{3'}
\end{equation}
then the operators $T$ and $S$ are called dual (in the sense of L. Zadeh). Obviously, if $T$ and $S$ satisfies (3'), then $S$ and $T$ also satisfies (3').

**Example 2.** Let us consider the Lukasiewitz t-norm $T_L$ with $T_L(x_1,x_2) = x_1 \oplus x_2 := \max(x_1 + x_2 - 1, 0)$, for all $x_1, x_2 \in [0,1]$ and the Lukasiewitz t-conorm $S_L$ with $S_L(x_1,x_2) = x_1 \otimes x_2 := \min(x_1 + x_2, 1)$, for all $x_1, x_2 \in [0,1]$. It is easy to check that they satisfy relation (3'). Hence they are dual operators, and for any $x = (x_1, x_2) \in L$ and any $y = (y_1, y_2) \in L$ the operation
\begin{align*}
(x_1, x_2) \oplus (y_1, y_2) &= (x_1 \oplus y_1, x_2 \otimes y_2) \\
(x_1, x_2) \otimes (y_1, y_2) &= (x_1 \oplus y_1, x_2 \otimes y_2)
\end{align*}
will be a (representable) t-norm and t-conorm on $L$, respectively. Clearly,
\begin{align*}
(x_1, x_2) \oplus (y_1, y_2) &= (\max(x_1 + y_1 - 1, 0), \min(x_2 + y_2, 1)) \tag{4} \\
(x_1, x_2) \otimes (y_1, y_2) &= (\min(x_1 + y_1, 1), \max(x_2 + y_2 - 1, 0)) \tag{4'}
\end{align*}

### 3 Rough sets and their representations

Rough sets were introduced by Z. Pawlak [14] in order to provide a formal approach to deal with incomplete data. In rough set theory, any set of entities is characterized by a lower approximation and an upper approximation.

Approximations are then defined in terms of an indiscernibility space, that is, a relational structure $(U, R)$ such that $R$ is a binary relation on $U$. In original definition of Pawlak, $R$ is an equivalence relation, but since inception, several generalizations of his construction had been proposed. Here we will consider $R \subseteq U \times U$ to be a quasiorder, i.e. a reflexive, transitive relation, since in this case the induced rough sets still form a completely distributive lattice (see Järvisi, Radeleczki and Veres [10]), similar to the case of an equivalence.

Let $R \subseteq U \times U$ be a quasiorder defined on the universe $U$. For any element $x \in U$ the set $R(x) = \{u \in U \mid (x, u) \in R\}$ is called the relational neighbourhood of $x$. Now, for any set $X \subseteq U$ its lower approximation and its upper approximation are defined as follows:

\begin{align*}
X_R := \{u \in U \mid R(u) \subseteq X\}, \quad X^R := \{u \in U \mid R(u) \cap X \neq \emptyset\}.
\end{align*}

The rough set of $X$ is the pair $(X_R, X^R)$, and the set of all rough sets is $RS = \{(X_R, X^R) \mid X \subseteq U\}$. In this approach, $X_R$ can be viewed as the set of elements
which certainly belong to \( X \), and \( X^R \) is interpreted as the set of objects that possibly are in \( X \), when elements are observed through the knowledge expressed by \((U, R)\). The set \( RS \) can be ordered by the coordinatewise order:

\[
(X_R, X^R) \leq (Y_R, Y^R) \iff X_R \subseteq Y_R \text{ and } X^R \subseteq Y^R.
\]

It was proved in Järvinen, Radeleczki and Veres [10] that the ordered set \( RS = (RS, \leq) \) is a completely distributive De Morgan algebra, and latter it was shown that a particular Kleene algebra can be defined on it (see Järvinen, Radeleczki [8]). A Kleene algebra is a De Morgan algebra \( \mathcal{K} = (K, \vee, \wedge, \sim, 0, 1) \) satisfying the (Kleene’s) axiom:

\[
x \wedge \sim x \leq y \vee \sim y, \text{ for all } x, y \in K.
\] (K)

A Heyting algebra \( L \) is a bounded lattice such that for all \( a, b \in L \), there is a greatest element \( x \in L \) satisfying \( a \wedge x \leq b \). This element \( x \) is called the relative pseudocomplement of \( a \) with respect to \( b \), and is denoted by \( a \Rightarrow b \). It is known that any completely distributive lattice is a Heyting algebra \( (L, \vee, \wedge, \Rightarrow, 0, 1) \) where the relative pseudocomplement is defined as follows:

\[
x \Rightarrow y = \bigvee \{ z \in L \mid x \wedge z \leq y \}.
\]

A Nelson algebra is a Kleene algebra \( \mathcal{N} = (N, \vee, \wedge, \sim, \Rightarrow, 0, 1) \) such that for any pair \( a, b \in N \) the relative pseudocomplement \( a \Rightarrow b \) of the element \( a \) with respect to \( \sim a \vee b \) there exists (see Cignoli [3]), and for any \( c \in N \) the equation:

\[
(a \wedge b) \Rightarrow c = a \Rightarrow (b \Rightarrow c).
\] (N)

holds. In each Nelson algebra, the weak negation \( \neg \) can be defined as

\[
\neg a := a \rightarrow 0, \text{ for all } a \in N.
\] (5)

Since \( RS \) is a completely distributive lattice, a Heyting algebra \( (RS, \vee, \wedge, \Rightarrow) \) can be defined on it. Moreover, in Järvinen, Radeleczki [8] it was proved that \( RS \) is a Nelson algebra.

Let \( \mathcal{P}(U)_R = \{ X_R \mid X \subseteq U \} \) and \( \mathcal{P}(U)^R = \{ X^R \mid X \subseteq U \} \). In [10] is proved that \( (\mathcal{P}(U)_R, \subseteq) \) and \( (\mathcal{P}(U)^R, \subseteq) \) are Heyting algebras. Clearly, \( (X_R, X^R) \) is an element of \( \mathcal{P}(U)_R \times \mathcal{P}(U)^R \). Let \( U - A \) stand for the complement of a set \( A \subseteq U \). It is known that \( U - X^R \) is just the set \( \langle U - X \rangle_R \in \mathcal{P}(U)_R \). Since each rough set is uniquely determined by the approximation pair \( (X_R, X^R) \), one can represent the rough set of \( X \) as a pair \( (X_R, U - X^R) = (X_R, \langle U - X \rangle_R) \in \mathcal{P}(U)_R \times \mathcal{P}(U)^R = (\mathcal{P}(U)_R)^2 \), too. Therefore, we construct the set

\[
DRS = \{ (X_R, U - X^R) \mid X \subseteq U \} \subseteq (\mathcal{P}(U)_R)^2,
\] (6)

which is ordered as follows:

\[
(X_R, U - X^R) \leq (Y_R, U - Y^R) \iff X_R \subseteq Y_R \text{ and } U - X^R \supseteq U - Y^R.
\]
Thus \((X_R, U - X^R) \leq (Y_R, U - Y^R) \iff (X_R \subseteq Y_R \text{ and } X^R \subseteq Y^R) \iff (X_R, X^R) \leq (Y_R, Y^R)\), and hence \(DRS\) and \(RS\) are order-isomorphic; therefore, they are isomorphic as Heyting algebras and as Nelson algebras, as well. Since \(X_R\) and \(U - X^R\) are disjoint sets, the above representation is called the disjoint representation of the rough sets of \(R\). The algebraic operations on \(DRS\) are defined for any \((A, B), (C, D) \in DRS\) as follows:

\[
(A, B) \lor (C, D) = (A \cup C, B \cap D), \\
(A, B) \land (C, D) = (A \cap C, B \cup D), \\
\sim (A, B) = (B, A)
\]

Because \(RS\) determines a Nelson algebra, for any pair \((X_R, X^R), (Y_R, Y^R) \in RS\) the Nelson implication \((X_R, X^R) \rightarrow (Y_R, Y^R)\) is also defined. By isomorphism to this operation \(\Rightarrow\) corresponds an operation \(\rightarrow\) on \(DRS\), expressed as

\[
(A, B) \rightarrow (C, D) := (A \Rightarrow C, A \cap D),
\]

where \(\Rightarrow\) is the Heyting operation of the algebra \((\mathcal{P}(U)_R, \lor, \land, \Rightarrow)\).

Now, we are closing this section by proving that the disjoint representations of rough sets of \(R\) can be interpreted as intuitionistic fuzzy sets. We will do this by identifying the disjoint representation \((X_R, U - X^R)\) of a rough set \((X_R, X^R) \in RS\) with the pair \(\left(\chi^{(X_R)}, \chi^{(U - X^R)}\right)\) formed by the characteristic function of the set \(X_R\), respectively \(U - X^R\). Observe that \(\left(\chi^{(X_R)}, \chi^{(U - X^R)}\right)\) is an intuitionistic set, since \(X_R, U - X^R \subseteq U\) and \(X_R \cap (U - X^R) \subseteq X^R \cap (U - X^R) = \emptyset\) imply that for any element \(x \in U\), only the following cases are possible:

1. \(x \in X_R\). Then \(x \notin U - X^R\), hence \(\chi^{(X_R)}(x) + \chi^{(U - X^R)}(x) = 1 + 0 = 1\),
2. \(x \in U - X^R\). Then \(x \notin X_R\), hence \(\chi^{(X_R)}(x) + \chi^{(U - X^R)}(x) = 0 + 1 = 1\),
3. \(x \notin X_R\) and \(x \notin U - X^R\). Then \(\chi^{(X_R)}(x) + \chi^{(U - X^R)}(x) = 0 + 0 = 0\).

Thus for any \(x \in U\) we get \(\chi^{(X_R)}(x) + \chi^{(U - X^R)}(x) \leq 1\).

## 4 Generalizing Nelson logic via intuitionistic fuzzy sets

Our aim is to generalize the notion of rough sets by using intuitionistic sets, in such a way to obtain algebraic structures similar to Nelson algebras, suitable to model similar logical rules. The significance of the Nelson algebras lies in the fact that they provide models for Constructive logic with strong negation (cf. Järvinen, Radeleczki [9]) introduced by D. Nelson. This logic is often called as Nelson logic. It is an extension of the intuitionistic propositional logic by strong negation \(\sim\), as shown by H. Rasiowa [15]. This logic is axiomatized by extending intuitionistic logic with the formulas (where \(p \leftrightarrow q = (p \rightarrow q) \land (q \rightarrow p)\)):

\[
\begin{align*}
(NL1) & \sim p \rightarrow (p \rightarrow q); \\
(NL2) & \sim (p \rightarrow q) \leftrightarrow p \land \sim q; \\
(NL3) & \sim (p \land q) \leftrightarrow \sim p \lor \sim q; \\
(NL4) & \sim (p \lor q) \leftrightarrow \sim p \land \sim q;
\end{align*}
\]
are always satisfied. We say that $A$ satisfies (NL5) \( \sim \sim p \leftrightarrow p \); (NL6) \( \sim p \leftrightarrow p \).

Since the rough sets defined by equivalence or quasiorder relations form Heyting algebras \((RS, \lor, \land, \rightarrow)\) which are very particular objects in fuzzy setting (see e.g. Novák, Perfilieva, & Mockor [13]), first we need to consider a notion which represents a generalization of Heyting algebras and it is used in the framework of fuzzy sets.

**Definition 1.** A bounded residuated lattice is an algebra $A = (A, \lor, \land, \ast, \rightarrow, 0, 1)$ of type $(2,2,2,2,0,0)$ such that

(a) \((A, \lor, \land)\) is a lattice with least element 0 and greatest element 1,
(b) \((A, \ast)\) is a commutative semigroup, such that $1 \ast x = x \ast 1 = x$, for all $x \in L$.
(c) $A$ satisfies the adjointness property, for all $x, y, z \in A$, that is

$$x \ast y \leq z \text{ if and only if } x \leq y \rightarrow z.$$  

We note the operation $i(x, y) = x \rightarrow y$ is order reversing in the first variable, and order preserving in its second variable, moreover, for all $x, y, z \in A$ the rules

$$x \ast (y \lor z) = x \ast y \lor x \ast z;$$

$$(x \ast y) \rightarrow z = x \rightarrow (y \rightarrow z)$$

$$x \leq y \leftrightarrow x \rightarrow y = 1$$

are always satisfied. We say that $A$ satisfies the double negation law if

$$(x \rightarrow 0) \rightarrow 0 = x,$$ \hspace{1cm} (DN)

for each $x \in A$. Denote $|x| := x \rightarrow 0$. Then (DN) means $|(|x|) = x$, and the map $x \mapsto |x|$ is a De-Morgan operation on the lattice $A$ (see e.g. Galatos N. et al. [5]).

Observe, that in fact any Heyting algebra is a particular residuated lattice with $\ast = \land$. It is also well-known that any continuous t-norm $T$ on a complete lattice $L$ induces a residuated structure on $L$ as follows:

$$x \rightarrow_T y := \sup \{z \in L \mid T(x, z) \leq y\}, \text{ for all } x, y \in L.$$

The implication $x \rightarrow y$ is called the residuated implication induced by $T$.

**Example 3.** (a) Consider the Lukasiewicz t-norm $x \oplus y = \max\{0, x + y - 1\}$, $x, y \in [0,1]$. Then $x \rightarrow_L y = \min\{1, 1 - x + y\}$. Clearly, $|x| = x \rightarrow 0 = 1 - x$, for all $x \in [0,1]$. Hence $|(|x|) = x$, for all $x \in [0,1]$. Thus $B = ([0,1], \max(\cdot), \min(\cdot), \oplus, \rightarrow_L, 0, 1)$ is a bounded residuated lattice satisfying (DN). It is well-known that $B$ satisfies the additional rules

$$x \oplus (x \rightarrow_L y) = \min(x, y) \text{ and } \max((x \rightarrow_L y), (y \rightarrow_L x)) = 1.$$  

Residuated lattices satisfying these rules and (DN) are called MV-lattices.

(b) Consider the implication induced by $T_L = \oplus$ defined on \( \{(x_1, x_2) \in [0,1]^2 \mid x_1 + x_2 \leq 1\} \) by (4). In Cornelis, Deschrijver and Kerre [4] it is shown, that

$$(x_1, x_2) \Rightarrow_L (y_1, y_2) := (\min(1, 1 - x_1 + y_1, 1 - y_2 + x_2), \max(0, y_2 - x_2)), \hspace{1cm} (7)$$

for all $x, y \in [0,1]$. It is easy to check that $(L, \lor, \land, \oplus, \Rightarrow_L, (0,1), (1,0))$ does not satisfy the double negation rule, because $|L(x_1, x_2) = (x_1, x_2) \Rightarrow_L (0,1) = (x_2, 1 - x_2)$, and hence it is not an MV-algebra.
Using the residuated implication $\rightarrow_L$ discussed in Example 3(a), we will define on the lattice $L$ of logical values a new binary operation $\rightarrow_L$:

$$(x_1, x_2) \rightarrow_L (y_1, y_2) := (x_1 \rightarrow_L y_1, x_1 \oplus y_2), \text{ for all } (x_1, x_2), (y_1, y_2) \in L. \quad (8)$$

Let $L$ be a lattice with least element 0 and greatest element 1. An implicator on $L$ is a mapping $i : L^2 \rightarrow L$ satisfying the conditions: $i(0, 0) = i(0, 1) = i(1, 1) = 1$ and $i(1, 0) = 0$. This notion is derived from fuzzy logic, where $L = [0, 1]$. An implicator $i$ satisfies the left neutrality principle, if $i(1, x) = x$, for all $x \in L$, and $i$ satisfies the identity principle, if $i(x, x) = x$, for all $x \in L$.

**Proposition 1.** The lattice $L$ is closed with respect to $\rightarrow_L$. The operation $\rightarrow_L$ is an implicator on $L$ which satisfies the left neutrality and identity principle.

**Proof.** First, we note the following: Because $([0, 1], \max(\cdot), \min(\cdot), \oplus, \rightarrow_L, 0, 1)$ is a residuated lattice with $[x = 1 - x$, for any $(x_1, x_2) \in L$ we have

$$x_1 + x_2 \leq 1 \Leftrightarrow x_1 \leq 1 - x_2 = x_2 \Leftrightarrow x_1 \oplus x_2 = 0.$$ 

Hence $(x_1, x_2) \in L$ if and only if $x_1 \oplus x_2 = 0$.

Now take any $(x_1, x_2), (y_1, y_2) \in L$. Then $x_1 \oplus x_2 = 0$ and $y_1 \oplus y_2 = 0$. Since by definition, $(x_1, x_2) \rightarrow_L (y_1, y_2) := (x_1 \rightarrow_L y_1, x_1 \oplus y_2)$, in order to prove $(x_1, x_2) \rightarrow_L (y_1, y_2) \in L$, it suffices to show that $(x_1 \oplus y_2) \oplus (x_1 \Rightarrow_L y_1) = 0$.

Because $\oplus$ is commutative, associative and order preserving, we obtain:

$$(x_1 \oplus y_2) \oplus (x_1 \Rightarrow_L y_1) = y_2 \oplus x_1 \oplus (x_1 \Rightarrow_L y_1) \leq y_2 \oplus y_1 = y_1 \oplus y_2 = 0.$$ 

This proves that $(x_1, x_2) \rightarrow_L (y_1, y_2) \in L$.

Next, recall that the least element and the greatest element in $L$ is $(0, 1)$, respectively $(1, 0)$. Now is a routine to check that

$$(0, 1) \rightarrow_L (0, 1) = (0, 1) \rightarrow_L (1, 0) = (1, 0) \rightarrow_L (1, 0) = (1, 0).$$

Thus $\rightarrow_L$ is an implicator on $L$. We get also

$$(1, 0) \rightarrow_L (x_1, x_2) = (1 \Rightarrow_L x_1, x_1 \oplus x_2) = (x_1, x_2),$$ 

meaning that $\rightarrow_L$ satisfies the left neutrality. Similarly, $(x_1, x_2) \rightarrow_L (x_1, x_2) = (x_1 \Rightarrow_L x_1, x_1 \oplus x_2) = (1, 0)$, hence the identity principle is also satisfied by $\rightarrow_L$.  

Finally, we introduce the notion of a *quasi Kleene algebra* which is defined as a De Morgan algebra $(K, \lor, \land, \oplus, \neg, 0, 1)$ extended with a strong conjunction $\ominus$ and a strong disjunction $\oplus$, and such that

$$\neg (x \oplus y) = (\neg x) \ominus (\neg y), \quad \neg (x \ominus y) = (\neg x) \ominus (\neg y) \quad (M^*)$$

$$x \ominus (\neg x) \leq y \ominus (\neg y), \text{ for all } x, y \in K. \quad (K^*)$$

Of course, replacing $\ominus$ by $\land$ and $\oplus$ by $\lor$ we reobtain as a particular case the notion of a Kleene algebra.

**Theorem 1.** The algebra $(L, \lor, \land, \oplus, \neg, (0, 1), (1, 0))$ is a quasi Kleene algebra, and the implicator $\rightarrow_L$ satisfies the identity:

$$a \ominus b \rightarrow_L c = a \rightarrow_L (b \rightarrow_L c). \quad (N^*)$$
Proof. Since \((\mathcal{L}, \lor, \land, \sim, (0, 1), (1, 0))\) is a De Morgan algebra, we have to check only \((M^*)\) \((K^*)\). Take any \(x = (x_1, x_2) \in \mathcal{L}\) and \(y = (y_1, y_2) \in \mathcal{L}\). By definition, \(\sim (x \oplus y) = \sim (x_1 \oplus y_1, x_2 \oplus y_2) = (x_2 \oplus y_2, x_1 \oplus y_1) = (x_2, x_1) \oplus (y_2, y_1) = (\sim x) \oplus (\sim y)\). The equality \(\sim (x \oplus y) = (\sim x) \oplus (\sim y)\) is proved dually. Thus \((M^*)\) is satisfied.

We get also \((x_1, x_2) \oplus (\sim (x_1, x_2)) = (x_1, x_2) \oplus (x_2, x_1) = (x_1 \ast x_2, x_2 \ast x_1) = (0, x_2 \ast x_1)\), because \(x_2 \leq 1 - x_1 \leq |x_1|\) implies \(x_1 \ast x_2 \leq x_1 * x_2 = x_1 * (x_1 \rightarrow L 0) = 0\) (by Def. \(1(c)\)). Similarly, we obtain \((y_1, y_2) \oplus (\sim (y_1, y_2)) = (y_1, y_2) \oplus (y_2, y_1) = (y_1 \oplus y_2, y_2 \oplus y_1) = (y_1 \oplus y_2, 0)\). Since \((0, x_2 \oplus x_1) \leq (y_1 \oplus y_2, 0)\), we deduce \(x \oplus (\sim x) = (x_1, x_2) \oplus (\sim (x_1, x_2)) \leq (y_1, y_2) \oplus (\sim (y_1, y_2)) = y \oplus (\sim y)\), proving that the algebra satisfies \((K^*)\).

In order to prove \((N^*)\), take any \(a = (a_1, a_2), b = (b_1, b_2), c = (c_1, c_2)\) such that \(a, b, c \in \mathcal{L}\). Then \((a \oplus b) \rightarrow_L c = (a_1 \oplus b_1, a_2 \oplus b_2) \rightarrow_L (c_1, c_2) = ((a_1 \oplus b_1) \rightarrow_L c_1, a_1 \oplus b_1 \rightarrow_L c_1)\). As \((a_1 \oplus b_1) \rightarrow_L c_1 = a_1 \rightarrow_L (b_1 \rightarrow_L c_1)\) holds in \([0, 1], \max(\cdot), \min(\cdot), \oplus, \rightarrow_L\), \(0, 1)\), we get \((a \oplus b) \rightarrow_L c = (a_1 \rightarrow_L (b_1 \rightarrow_L c_1), a_1 \oplus b_1 \rightarrow_L c_1)\) in \(\mathcal{L}\).

Since the right side of the equality in \((N^*)\) yields \(a \rightarrow_L (b \rightarrow_L c) = (a_1, a_2) \rightarrow_L ((b_1, b_2) \rightarrow_L (c_1, c_2)) = (a_1, a_2) \rightarrow_L (b_1 \rightarrow_L c_1, b_1 \oplus c_1) = (a_1 \rightarrow_L (b_1 \rightarrow_L c_1), a_1 \oplus b_1 \rightarrow_L c_1)\), we obtain that \((a \oplus b) \rightarrow_L c = a \rightarrow_L (b \rightarrow_L c)\). \( \square \)

Corollary 1. The intuitionistic sets on \(U\) form a quasi Kleene algebra \((\mathcal{I}(U), \lor, \land, \oplus, \sim, (0, 1), (1, 0))\).

Finally, we will prove that on \(\mathcal{L}\) can be defined a propositional calculus that satisfies almost all additional logical rules \((NL1) - (NL6)\) used to extend the intuitionistic logic into Nelson logic. In order show this, it suffices to prove that \(\mathcal{L}\) satisfies some identities corresponding to the tautologies \((NL1) - (NL6)\). More precisely, defining \(x \leftrightarrow y = (x \rightarrow_L y) \land (y \rightarrow_L x)\), for all \(x, y \in \mathcal{L}\), we can prove

**Theorem 2.** The algebra \(\mathcal{A} = (\mathcal{L}, \lor, \land, \oplus, \sim, \rightarrow_L, (0, 1), (1, 0))\) satisfies the identities: \n
\begin{align*}
(NL1) & \sim x \rightarrow_L (x \rightarrow_L y) = (1, 0); \\
(NL2') & \sim (x \rightarrow_L y) \leftrightarrow (x \oplus \sim y) = (1, 0); \\
(NL3) & \sim (x \land y) \leftrightarrow (\sim x \lor \sim y) = (1, 0); \\
(NL3') & \sim (x \oplus y) \leftrightarrow (\sim x \oplus \sim y) = (1, 0); \\
(NL4) & (x \lor y) \sim (\sim x \land \sim y) = (1, 0) \\
(NL4') & (x \oplus y) \sim (\sim x \oplus \sim y) = (1, 0); \\
(NL5) & \sim x \leftrightarrow x = (1, 0); \\
(NL6) & \sim x \rightarrow_L x = (1, 0),
\end{align*}

**Proof.** Since \(\sim\) satisfies the identity principle, we have \(x \leftrightarrow x = (x \rightarrow_L x) \land (x \rightarrow_L x) = (1, 0) \land (1, 0) = (1, 0)\). Therefore, in the case when \(A = B\) is an
Hence for all \((NL6)\). Here we recall that for any identity in \((NL3),(NL3'),(NL4), (NL4'), (NL5)\) hold for all \(x, y \in L\).

\((NL6)\). Here we recall that for any \(x = (x_1, x_2) \in L\), we have by definition
\[
\sim x =: (x_1, x_2) \rightarrow_L (0, 1) = (x_1 \rightarrow_L 0, x_1 \oplus 1) = (1 - x_1, x_1),
\]
therefore
\[
\sim \sim x = (x_1, 1 - x_1).
\]

Hence for all \(x \in L\) we obtain:
\[
\sim x \leftrightarrow x = ((x_1, 1 - x_1) \rightarrow_L (x_1, x_2)) \land ((x_1, x_2) \rightarrow_L (x_1, 1 - x_1)) =
= (x_1 \rightarrow_L x_1, x_1 \oplus x_2) \land (x_1 \rightarrow_L x_1, x_1 \oplus (1 - x_1)) = (1, 0) \land (1, 0) = (1, 0),
\]
and this proves \((NL6)\).

\((NL1)\). As \(L\) satisfies \((N^*)\), for any \(x = (x_1, x_2) \in L\), \(y = (y_1, y_2) \in L\) we get:
\[
\sim x \rightarrow_L (x \rightarrow_L y) = (\sim x \odot x) \rightarrow_L y = ((x_1, x_1) \odot (x_1, x_2)) \rightarrow_L (y_1, y_2) =
= (x_2 \odot x_1, x_1 \odot x_2) \rightarrow_L (y_1, y_2) = (0, x_1 \odot x_2) \rightarrow_L (y_1, y_2) = (0 \rightarrow_L y_1, 0) = (1, 0),
\]
because \(0 \rightarrow_L y_1 = 1\).

\((NL2')\). Take any \(x, y \in L\) with \(x = (x_1, x_2)\) and \(y = (y_1, y_2)\). Then
\[
\sim (x \rightarrow_L y) \rightarrow_L (x \odot x \sim y) = \sim (x_1 \rightarrow_L y_1, x_1 \odot y_2) \rightarrow_L ((x_1, x_2) \odot (y_2, y_1)) =
= (x_1 \odot y_2, x_1 \rightarrow_L y_1) \rightarrow_L (x_1 \odot y_2, x_2 \odot y_1) =
= ((x_1 \odot y_2) \rightarrow_L (x_1 \odot y_2), x_1 \odot y_2 \oplus y_2 \oplus y_1) = (1, 0)
\]
(since \(\rightarrow_L\) satisfies the identity principle and \(x_1 \odot x_2 = y_2 \odot y_1 = 0\).

Similarly, we obtain:
\[
(x \odot y) \rightarrow_L \sim (x \rightarrow_L y) = (x_1 \odot y_2, x_2 \odot y_1) \rightarrow_L (x_1 \odot y_2, x_1 \rightarrow_L y_1) =
= (x_1 \odot y_2 \rightarrow_L (x_1 \odot y_2), x_1 \odot y_2 \oplus (x_1 \rightarrow_L y_1)) =
= (1, y_2 \oplus (x_1 \odot (x_1 \rightarrow_L y_1))) = (1, 0),
\]
because
\[
x_1 \odot (x_1 \rightarrow_L y_1) \leq y_1 \text{ implies } y_2 \odot (x_1 \odot (x_1 \rightarrow_L y_1)) \leq y_2 \odot y_1 = 0.
\]

Thus we deduce \((NL2')\): \(\sim (x \rightarrow_L y) \leftrightarrow (x \odot y) = (1, 0) \land (1, 0) = (1, 0)\)

5 Conclusions

In the previous sections we have shown that there are several analogies between the intuitionistic fuzzy sets and rough sets.

First, we proved that the rough sets induced by a quasiorder relation \(R\) can be viewed as particular intuitionistic sets. Moreover, rough sets induced by \(R\) form particular Kleene algebras, and analogously, the intuitionistic sets on a universe \(U\) form quasi Kleene algebras.

Secondly, in a logical approach, both rough sets and both the intuitionistic sets are tools for handling the incomplete information, see e.g. Kovács, Radeleczki [11]. Rough sets induced by a quasiorder are fundamental models for Constructive logic with strong negation (or Nelson logic) which is obtained by extending the intuitionistic propositional logic with some formulas involving the strong negation \(\sim\) (see e.g. Järvinen, Radeleczki [9]). We note that the algebraic counterpart of the intuitionistic logic is formed by Heyting algebras. Similarly, for any continuous t-norm \(T\) defined on the lattice \(L = \{(x_1, x_2) \in [0, 1]^2 \mid x_1 + x_2 \leq 1\}\) of logical values a residuated lattice \((L, \vee, \wedge, T, \rightarrow_T, (0, 1), (1, 0))\) can be constructed; the residuated lattices constitute a natural generalization of Heyting algebras. It is known that the logics
defined on several types of residuated lattices belong to the family of the so-called substructural logics (see Galatos et al. [5], or Novák,Perfilieva & Mockor [13]). Therefore, a connective system defined on $\mathcal{L}$ leads to a particular substructural logic. We proved that in the case of Lukasiewitz t-norm $\odot$, this logic can be extended by adding to it almost the same rules involving the standard negation $\sim$ as in the case of Nelson logic. In fact, all the particular rules of this logic remain valid in the framework of $\mathcal{L}$, except (NL2), which is replaced by the rule

\[(NL2') \sim (p \rightarrow_L q) \leftrightarrow (p \odot \sim q)\]

Hence in the case of the intuitionistic sets and their logic, the standard negation $\sim (x_1, x_2) = (x_2, x_1)$ has the same role as the strong negation in Nelson logic.

References