# FROM THE THEOREM OF THE BROKEN CHORD TO THE BEGINNING OF TRIGONOMETRY


**Maria Drakaki**

Doctoral Student

Department of Mathematics and Applied Mathematics

University of Crete

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## ABSTRACT

The subject of this paper is to present Archimedes' Broken Chord Theorem (3rd century BC) and its relationship to the origins of Trigonometry. We learn about this theorem from the medieval Arab mathematician Al Biruni in his treatise entitled *Book on the Derivation of Chords in a Circle*, which the Swiss Heinrich Suter (1848-1922) translated into German (Zurich, 1910). In this work of Al Biruni, we can find 22 proofs of the Broken Chord Theorem, amongst which three are attributed to Archimedes.

Al Biruni's methods follow Ptolemy in the Almagest, where he has an elaborate method of calculating the table of chords. In turn Ptolemy uses ideas of Hipparchus who also constructed a table of chords which, however, is lost today. An analysis of techniques from antiquity shows that Archimedes' Broken Chord Theorem leads to the same steps. In fact we can recognize the Broken Chord Theorem and other results of Archimedes in a proof of a theorem in the Almagest. The Broken Chord Theorem served Archimedes in his studies of Astronomy as an analogous formula to our $sin(x-y) = sinxcosy - cosxsiny$ and since this formula gives the ability to construct a chord table, we can assume that Archimedes was in possession of one of these.

So, in Archimedes' work we detect the first glimmers of Trigonometry and we have every reason to ask ourselves: Is Archimedes the founder of Trigonometry?



**Keys**

The Theorem of the Broken Chord, Chord table, Origins of Trigonometry

# Contents

# Chapter 1

# The Broken Chord Theorem

## 1.1 Introduction

Carl Schoy[1] (1877-1925), shortly before his premature death which deprived the science of mathematics history of his promising work, had translated from Arabic some excerpts from the great trigonometry work *Al-Qânûn Al-Mas'ûdi* (1030) of the Iranian scholar Al Biruni [2].

Figure 1.1: Al Biruni

These excerpts, together with some other important similar texts, were published by J. Ruska and H. Wieleitner just two years after his death[3]. C. Schoy was regarded as the successor of the Swiss Heinrich Suter (1848-1922), who was a connaisseur of Arabic and expert in Arabic Mathematics. H. Suter had translated (into German) and commented on the work of Al Biruni *Book on the Derivation of Chords in a Circle*. If we study this work of Suter in combination

[1] A German source researcher and author in the field of Arabic Astronomy, Gnomonik and Mathematics.

[2] Iranian polymath scholar (973-1048),who was well versed in Physics, Mathematics, Astronomy and distinguished himself as a historian and linguist.

[3] C. Schoy, *Die trigonometrischen Lehren des persischen Astronomen Al Biruni*, Hannover 1927.

with the aforementioned work of Schoy, some very interesting and not widely known conclusions emerge, regarding the early history of Trigonometry and in particular the contribution of Archimedes (287-212 BC) to it [4].

It is surprising in these two works the discovery of a work of Arhimedes on the heptagon, which was considered lost since the Era of the Arabs. There was a translation into Arabic by Tabit ibn Qurra, who had in front of him the original, but quite ruined manuskript of Archimedes. Thanks to Schoy, this work has been translated into German.

## 1.2 The Broken Chord Theorem

In both his works (*Al-Qânûn* and *Book on the Derivation of Chords in a Circle*) Al Biruni formulates the Broken Chord theorem, whose paternity is best attributed to Archimedes. In the *Book on the Derivation of Chords in a Circle* he also cites 22 proofs mentioning the author of each of them. The same theorem uses the Persian Al-Shanni in his proof for calculating the area formula for the triangle area in terms of its three sides. The "Broken Chord theorem" or "Broken line theorem" is as follows:

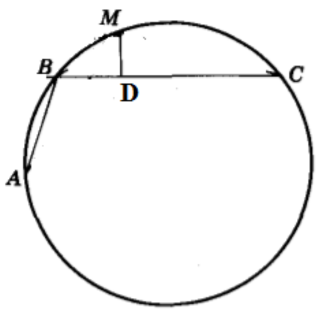



Figure 1.2:

If AB and BC make up a broken chord in a circle, where BC > AB, and if M is the midpoint of arc ABC, then the foot D of the perpendicular from M on BC is the midpoint of the broken chord ABC.

That is $AB + BD = DC$.

First Al Biruni mentions the proof of Archimedes followed by a similar of the Arab Djiordjani. Then there are two more proofs of Archimedes, which together with the first one, are in his treatise entitled *On touching circles*[5]. The remaining proofs are attributed either to himself or to Arabs, such

[4] Johannes Tropfke, *Archimedes und die Trigonometrie*, *Archiv für die Geschichte der Mathematik*, 10 (1927-1928), pages 432-463.

[5] Archimedes, *On touching circles*, proposition 14, pages 12-15.

as Hashbas, Sidjzi, Shanni, Iraq and others.

## 1.3 Some proofs of the Theorem

Let's see some proofs of the theorem, starting with those of Archimedes.

**Proof 1 (Archimedes)**

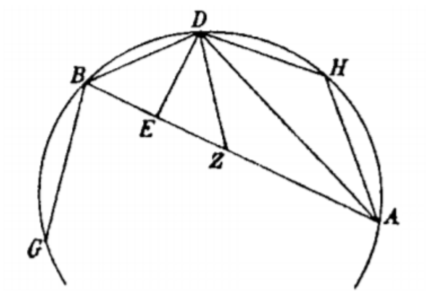


Figure 1.3:

Let D be the midpoint of the arc $\widehat{ABG}$, DH=DB and EZ=EB, with E the foot of the perpendicular from D on BA. We draw DZ, DA and we have that DB=DZ, since DE is altitude and median in the triangle BDZ. So DH=DZ. Since $\widehat{DB} = \widehat{DH}$ we have also $\widehat{AH} = \widehat{BG}$. So $\widehat{HDA} + \widehat{DAB} = \widehat{DBA} = \widehat{DZB}$. But $\widehat{DZB} = \widehat{ZAD} + \widehat{ZDA}$. It follows that $\widehat{ZDA} = \widehat{HDA}$. So, we can conclude that the triangles DZA, DHA are equal (since DH=DZ, DA=DA and $\widehat{ZDA} = \widehat{HDA}$) and hence[6] AZ=AH. But AH=BG and EZ=EB, so we have our final statement that AZ+EZ=EB+BG.

**Proof 2 (Archimedes)**

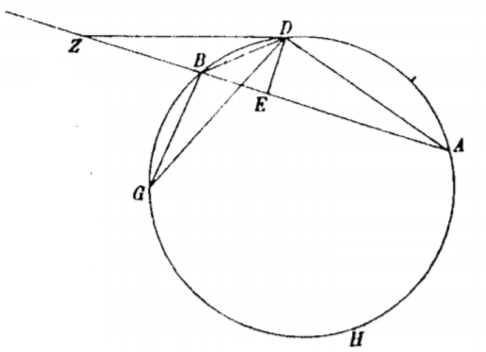


Figure 1.4:

We extend the AB, make EZ = AE and draw the DA, DG, DB, and DZ. Since the chords AD, DG are equal and AD=DZ, DZ=DG and since $\widehat{DAB} = \widehat{DGB}$, $\widehat{DAE} = \widehat{DZE}$, it follows that $\widehat{DZB} = \widehat{DGB}$. Since $\widehat{DA} = \widehat{DG}$ and the arc $\widehat{AHG}$ is in common, we have $\widehat{DAHG} = \widehat{DGHA}$.

[6]Archimedes does not refer to the equality of triangles, but goes directly to the equality of the other elements resulting from it.

But the $\widehat{DBG}$ is subtended by the $\widehat{DAHG}$, while $\widehat{DAB}$ and $\widehat{ADB}$ additionally are subtended by the $\widehat{DGHA}$. So, $\widehat{DBG} = \widehat{DAB} + \widehat{ADB}$. But the $\widehat{DBZ}$ is an exterior angle of the triangle DBA, then $\widehat{DBZ} = \widehat{DAB} + \widehat{ADB}$. Thus $\widehat{DBZ} = \widehat{DBG}$. We have prooved though that $\widehat{DZB} = \widehat{DGB}$, so it follows that $\widehat{DGB} = \widehat{ZDB}$. Finally, the triangles DZB, DBG are equal and hence BG=BZ. So we have the final statement that BG+BE=ZE=AE.

The third proof, which is just a variation of the previous one, was found by Al Biruni also in a second Greek paper entitled *Problems of the Greeks*[7] that existed at the time in an arabic translation by the Jûhanna b. Jûsuf (around 980/81) and for which Apollonius[8] was considered as its author.

**Proof 3 (Archimedes)**

(Figure 1.4)

Since the arc $\widehat{DBG}$ is less than the semicircle, it follows that the angle $\widehat{DBG}$ is obtuse and since the arc $\widehat{DGA}$ is more than the semicircle, it follows that the $\widehat{DBA}$, which is opposite of the chord AD, is an acute one and hence the $\widehat{DBZ}$ is obtuse. Furthermore, $\widehat{DZB} = \widehat{DGB}$ and DG=DZ too, thus $\dfrac{DG}{DB} = \dfrac{DZ}{DB}$.

The triangles DBZ and DBZ have an equal angle and the sides that contain the angles $\widehat{DBG}$ and $\widehat{DBZ}$ proportional, so since these are obtuse, they are equal and the triangles are similar. But they are equal too, therefore BZ = BG, ie BG + BE = AE, as required.

The shortest proof in this work of Al-Biruni is attributed to As-Sidjzi (972) and is as follows:

[7] This work is not known to us today, neither in Greek nor in Arabic.

[8] Apollonius Pergaeus (late 3rd – early 2nd centuries BC) was a Greek Geometer und Astronomer known for his theories on the topic of conic sections.

Proof 4 (As-Sidjzi)

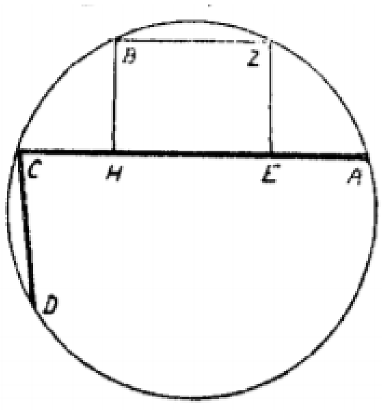



Figure 1.5:

Let B be the midpoint of the arc DBA, in which the broken line ACD is contained. We draw the BZ||CA and ZE||BH. Then, the arcs CB, ZA are equal, CD=BZ too and the proof goes on as in proof 2.

A very elegant proof was given by Gregg Patruno, a student at the Stuyvesant High School in New York[9].

Proof 5 (Patruno)

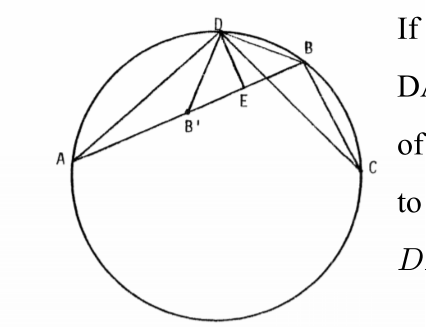



Figure 1.6:

If D is the midpoint of the arc $\widehat{ABC}$, we will have DA = DC and since $\widehat{DAB} = \widehat{DCB}$, then a rotation of the DBC triangle around D will bring the point C to A and B to a point $B'$ on the AB. Now the Triangle $DBB'$ is isosceles , so $EB = EB'$ and

$$AE = EB' + AB' = EB + BC$$

.

[9]*Crux Mathematicorum*, Volume 6, June-July 1980, page 189

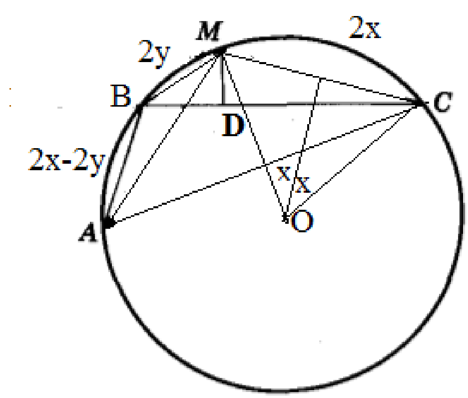
# Chapter 2

# Archimedes and Trigonometry

## 2.1 The trigonometric significance of the Theorem of the Broken Chord

We do not know if Archimedes saw any trigonometric significance in this theorem, but it has been written[1] that it served for him in his studies in astronomy as an analogous formula to our

$$sin(x - y) = sinxcosy - cosxsiny$$




Actually, if we put $\widehat{MC} = 2x$ and $\widehat{B\mathrm{M}} = 2y$, then $\widehat{AB} = 2x - 2y$.

Thus for a unit circle, the chords corresponding to these arcs are respectively MC=2sinx, BM=2siny, AB=2sin(x-y). Equally, the projections of MC and MB on BC are DC = 2sinxcosy and DB = 2sinycosx. Writing the Broken chord theorem in the form AB = DC-DB and

[1] Johannes Tropfke, *Archimedes und die Trigonometrie*, *Archiv für die Geschichte der Mathematik*, 10 (1927-1928), pages 432-463.

replacing the chords with their trigonometric equivalents, arises the type

$$sin(x-y) = sinxcosy - cosxsiny$$

Other trigonometric identities can also be derived from the same theorem. Since BC=BD+DC we have also

$$sin(x+y) = sinxcosy + cosxsiny$$

If we replace $\hat{x}$ with $\widehat{90^o - x}$ arise the analogous formulas for the cosines, i.e

$$cos(x+y) = cosxcosy - sinxsiny$$

and

$$cos(x-y) = cosxcosy + sinxsiny.$$

We therefore see that Archimedes' proposition completely replaces Ptolemy's proposition with a single disadvantage, perhaps, that in the context of the Greek theory of chords one must consider constructing another circle on the BMD triangle, so that we can conclude the relationship DB=BMcosx (or DB=CBsin(90-x)). Similarly for the triangle MDC.

## 2.2 The sine of half the angle

Another trigonometric formula attributed to Ptolemy which Archimedes certainly knew is the formula

$$sin\frac{a}{2} = \sqrt{\frac{1-cosa}{2}}.$$

Ptolemy did not usually refer to his predecessors - especially when their works were lost. Tropfke presents [2] the work of Archimedes on the same formula and compares

[2]Johannes Tropfke, *Archimedes und die Trigonometrie*, *Archiv für die Geschichte der Mathematik*, 10 (1927-1928), pages 432-463.

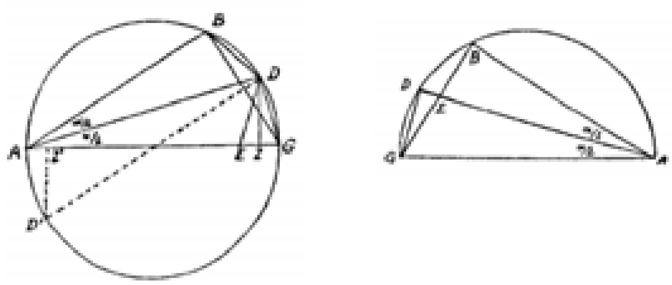


Figure 2.1:

it with that of Ptolemy, proving that Archimedes knew this rule 4 centuries earlier.

Let's see how Archimedes came to this formula. In his work *Measurement of a circle*[3] Archimedes calculates the perimeter of regular polygons with number of sides n = 6, 12, 24, 48 and 96 that are circumscribed and inscribed in a circle. We are interested in the inscribed ones, because their sides correspond to chords of $60^o, 30^o, 15^o, 7^o30'$, and $3^o45'$ respectively. It was at this point that Archimedes had to calculate half the angle, given the whole, something that Ptolemy also had to do four centuries later.

The figure used by Ptolemy (Figure 2.1, left) is the same (even in the choice of letters) as that of Archimedes (Figure 2.1, right), with the only difference that one figure is the mirror image of the other.

The calculation of this sine is also present in Archimedes' work on the heptagon[4] and in particular, it is the proposition 14 in Schoy's translation based on the work of Tabit ibn Qurra. The figure used by Tabit is again what we see on the right in Figure 2.1, with the only difference being the alternation of letters B and G. So let's look at the proposition 14 and its proof, according to Tabit's translation[5].

Proposition 14: Let ACB be a semicircle of center Z, AB=d its diameter, AC a chord

[3] *Archimedes, Opera omnia*, ed. Heiberg, Volume I, page 232.
[4] We referred to this work in paragraph 1.1.
[5] Tropfke, *Die Siebeneckhandlung des Archimedes*, Osiris, Vol. 1, (Januar 1936), pp 636-651.

and D the midpoint of the arc BC. We draw the DB, DC and make AH=AC. We claim that $ZB \cdot BH = BD^2$

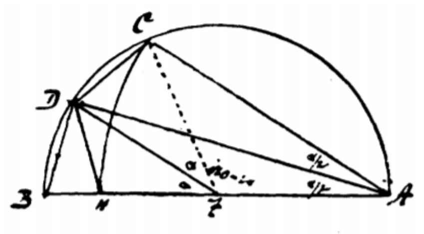


Figure 2.2:

Proof

The triangles CDA, HDA are equal, so DC=DH=DB.

The triangles BHD, BDZ are similar, so we have

$$\frac{HB}{BD} = \frac{BD}{DZ}$$

i.e $BD^2 = HB \cdot DZ = HB \cdot BZ$.(*)

According to the ancient chord trigonometry we have:

$BZ = \frac{1}{2}d$

$BD = d$·chord(α), meaning the chord corresponding to the central angle α

$AH = CA = d$·chord(180-2α)

$HB = AB - HA = d - d$· chord(180-2α)

The formula (*) is written as follows:

$$(d.chord(\alpha))^2 = \frac{1}{2}d \cdot (d - dchord(180 - 2\alpha))$$

If we write this relationship with modern symbolism, emerges the following :

$$(sin\frac{a}{2})^2 = \frac{1 - cosa}{2}$$

as required.

## 2.3 Is Archimedes the founder of Trigonometry?

Archimedes in this proposition shows that he has the theoretical background for calculating the x ±y chords, when the chords x, y are given. Ptolemy's proposition is clearly an improvement on Archimedes' proposition. But if it was truly an achievement of Ptolemy, who lived in Alexandria in the 2nd century AD, we would have to draw the conclusion that Hipparchus (in the middle of the 2nd century BC), to which we usually look for when we seek the roots of chords' theory, but also Menelaus[6] (98 AD) had used Archimedes' proposition in their own chord calculations. Especially Hipparchus could have known this new theory since he is chronologically right after Archimedes.

However, we should note that this proposition had neither been mentioned, nor used trigonometrically in any ancient mathematical work, until the time of Al-Biruni. Only in two Greek papers appears a similar figure: in proposition 3 in Archimedes's work *Liber assumptorum*[7] and in Ptolemy's *Almagest* [8], by his calculating of the half-angle chord given the whole one. If Hipparchus or Menelaus had used Archimedes' proposition, one would have had to assume that the proposition had become known in its full form. However, since this had not happened, it is possible that Hipparchus was already aware of Ptolemy's proposition and its use in chord theory. So the basis, that Archimedes might have put with his proposition for Trigonometry was forgotten, just as with all the works before Euclid, after the emergence of Euclid's *Elements*.

## 2.4 "Who is the founder of Trigonometry?" an open question

Researching the origins of Trigonometry on a personal level, I studied the views of various authors of the last three centuries and researched well-known and lesser-known works. In this section I will briefly mention some of them and refer to their respective works for more details.

[6]Greek mathematician and astronomer who first conceived and defined a spherical triangle.
[7] *Archimedes, Opera omnia*, ed. Heiberg, Bd II, Seite 513.
[8]*Cl.Ptolemaei Opera quae exstant omnia*, ed Heiberg, Leipzig 1898-1903, Bd I, Seite 39.

1893: Paul Tannery in his *Recherches sur l'histoire de l'astronomie* refering to Hipparchus claims that the invention of Trigonometry is not in fact the work of one. Hipparchus found prepared ground due to Archimedes and Apollonius. [13]

1900: Anton von Braunmühl writes:"With the astronomer Aristarch of Samos, who lived around 270 BC, we encounter a weak attempt to calculate trigonometric relationships mathematically...but only with Hipparchus, trigonometry could be considered a true science." [3]

1910: Suter translated the work of Al-Biruni *Book on the Derivation of Chords in a Circle*. [12]

1921: Heath refers to Tannery and writes that Apollonius, or Archimedes before him may have compiled a "table of chords", or at least shown the way to such a compilation,... but this is, however, in the region of conjecture. The first person to make systematic use of trigonometry is, as far as we know, Hipparchus. [5]

1928: Johannes Tropfke in his article entitled *Archimedes und die Trigonometrie* comes to the conclusion: "What Hipparchus achieved in Trigonometry, refers back to Archimedes' genius". [15]

1928: G.A. Miller in his article entitled *Archimedes and Trigonometry* in the journal "Science" claims that "..now we have more substantial reasons for regarding Archimedes as the founder of trigonometry.." and he refers to the discoveries about the formula $sin\frac{a}{2} = \sqrt{\frac{1-cosa}{2}}$ which Archimedes knew as we have seen. [8]

1933: Rome in his article *Premiers essais de trigonométrie rectiligne chez les grecs* writes: "Before we were able to build chord tables, we had to try to calculate them little by little. There are already such efforts in Aristarchus of Samos, in Euclid himself, and above all, in Archimedes". Rome took into account the works of Tannery, Suter, Schoy, Tropfke and refers to the calculation of the $sin\frac{a}{2}$ from Archimedes and to the theorem of Broken Chord, which allowed to calculate the x ±y chords, when the chords x, y are given. [9]

1966: Becker writes that Hipparchus had in his possession a chord table for his astronomical calculations, but..“we know now due to Al Biruni, that Archimedes knew and proved the theorem about the $sin(x + y)$, while his predecessor Aristarchos did not seem to possess such a trigonometrical table...” [1]

1968: Boyer in his work *A History of Mathematics*, in the Chapter X entitled “Greek Trigonometry and Mensuration”, writes:.. “In the works of Euclid there is no trigonometry in the strict sense of the word, but there are theorems equivalent to specific trigonometric laws or formulas. Propositions II. 12 and 13 of the *Elements*, for example, are the laws of cosines for obtuse and acute angles respectively....We have seen also that Archimedes' theorem on the Broken chord can readily be translated into trigonometric language analogous to formulas for sines of sums and differences of angles...”. Boyer claims that Hipparchus earned the right to be known as “the father of trigonometry”, since he apparently compiled the first trigonometric table. [2]

1973: Toomer writes: “My conclusion, that the plane trigonometry of the *Almagest* owes much to Ptolemy, and that the trigonometry of his predecessors, notably Hipparchus, was a great deal cruder, is contrary to the opinion of most who have discussed this subject. I do not wish to argue here against the views of Tannery, who maintained that the chord table of Hipparchus was essentially the same as Ptolemy's, and that there was a pre-Hipparchian chord table, based in part on the work of Archimedes and Apollonius, from which the Indian sine table was derived. These views were not supported by any solid arguments, though Tannery, who “had a nose”, as the French say, divined the true historical sequence; but he assigned the steps to the wrong authors, misled, as often, by the unjustified equation “Almagest” = “Hipparchus”. I am more concerned to refute the views of Tropfke. In his article “Archimedes und die Trigonometrie”, valuable in many ways, and particularly for calling attention to the sources for Archimedes’ work preserved only in Arabic, he suggests that Archimedes founded Greek trigonometry”. Toomer then presents his arguments and concludes that “.. though Archimedes’ work was important as a basis for later developments, we cannot attribute trigonometry to him. The credit for the creation of Greek trigonometry must go to Hipparchus, and for its improvement to the “classical” forms of antiquity to Ptolemy”. [14]

2009: Glen Van Brummelen in his book *The Mathematics of the Heavens and the Earth* looks at the controversies as well, including disputes over whether Hipparchus was indeed the father of trigonometry. He writes “Quantitative mathematical astronomy in ancient Greece did not begin overnight; its origins may be identified with several possible candidates. Two of the earliest are Aristarchus of Samos (ca. 310 BC – 230 BC) and Archimedes of Syracuse (287 BC – 212 BC).....The extent to which Archimedes was able to practice trigonometry itself is a matter for interpretation and debate. His uses of the lemma are ingenious, but they are qualitatively different from the systematic abilities to manipulate arcs and lengths that we find later”. Then he mentions the Theorem of Broken Chord and its trigonometrical significance, but he adds that.. “However, we have no evidence that trigonometry was Archimedes’ intent for this result; we may only assert that if he did turn his efforts in a trigonometric direction, he would have had the mathematics at his disposal to pull it off” [4].

## 2.5 Conclusion

The question about the founder of Trigonometry is still open. The most text-books on this subject refer to the Greek Astronomer Hipparchus, who is the first person, who made systematic use of trigonometry. However we can not ignore the achievements of Archimedes, which give us substantial reasons for regarding Archimedes as the founder of trigonometry.

Research will continue!

# Bibliography


[1] Becker, Oskar, *Das mathemathische Denken der Antike*, VIII, S. 110, Göttingen, 1966.

[2] Boyer, Carl, *A History of Mathematics*, Wiley editions, USA, 1968.

[3] Von Braunmühl, Anton, *Vorlesungen über Geschichte der Trigonometrie*, Teubner Verlag, Leipzig, 1900.

[4] Van Brummelen, Glen, *The Mathematics of the Heavens and the Earth: The Early History of Trigonometry*, Princeton, 2009.

[5] Heath, Thomas, *A History of Greek Mathematics*, Vol. II, XVII, Oxford, 1921.

[6] Heiberg J.L, *Archimedis Opera omnia:cum commentariis Eutocii.*, Bd II, Teubner Verlag, Leipzig, 1880.

[7] Heiberg J.L, *Cl.Ptolemaei Opera quae exstant omnia*, Bd I, Leipzig 1898-1903.

[8] Miller, G.A, Archimedes and Trigonometry, *Science, Vol. 67, June 1928, page 555.*

[9] Rome, A.,Premiers essais de trigonométrie rectiligne chez les grecs, *L'antiquite classique*, Tome 2, 1933, pp 177-192. classique, 2, 177-192.

[10] Schoy, Carl, *Die trigonometrischen Lehren des persischen Astronomen Al Biruni*, Hannover 1927.

[11] Scriba, C.J, Schreiber, P, *5000 Jahre Geometrie*, Springer editions, 2010, 78-79.

[12] Suter, Heinrich, Das Buch der Auffindung der Sehnen im Kreise von Abul-Kaihän Muh. el-Biruni, *Zeitschrift für Geschichte der Mathematischen Wissenschaften*, Teubner, Leipzig 1910-1911.

[13] Tannery, P, *Recherches sur l'histoire de l'astronomie ancienne*, Paris, 1893.

[14] Toomer G.J, *The Chord Table of Hipparchus and the Early History of Greek Trigonometry*, Centaurus, Vol. 18, Issue 1, p.6-28, 1974.

[15] Tropfke, Johannes, Archimedes und die Trigonometrie, *Archiv für die Geschichte der Mathematik, der Natunwissenschaften und der Technik*, 10, 1928, 432-461.

[16] *Crux Mathematicorum*, Volume 6, June-July 1980.