Riemann Sums Belong at the End of Integral Calculus; Not at the Beginning

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Introduction

Given the documented difficulties with the teaching of calculus, there have been a number of reforms suggested over the years. Recently, there has been a suggestion to reorder the topics in the calculus sequence to present integral calculus before differential calculus, citing its historical precedence [Bressoud, 2019]. This idea certainly has merit and deserves consideration, but it is not new. For example, the calculus book by Tom Apostol [1961] presents the concepts and applications of integral calculus before proceeding to definitions of limits, continuity, and finally differentiation and the Fundamental Theorem of Calculus.

These suggestions are certainly interesting and follow the historical evolution of the subject. However, there may be some concern about rearranging the entire calculus sequence in this manner. This presentation proposes that it is possible to utilize the overall historical development of the subject, while staying within the confines of the current sequential structure of the curriculum adopted by the majority of institutions teaching calculus. This can be done in the first semester differential calculus course, but here we will focus on the second semester integral calculus course. So for this approach, it is assumed that a student has successfully completed a first semester course in differential calculus.

Using Historical Development as a Pedagogical Context

"Ontogeny recapitulates phylogeny" is a debunked biological theory that a developing embryo goes through stages resembling the evolution of the species. As much as this is discredited in biology, there is possibly some merit to this in education theory. Indeed, an argument for mathematics teachers learning the history of mathematics is not only adding a humanistic aspect to mathematics, but also learning to recognize alternate approaches to problem solving. Since often the study of the history of mathematics focuses on the development of tools, techniques, and language to solve specific problems, then it is reasonable to assume that students will reproduce some of these techniques and strategies in developing their understanding. Starting with abstract concepts and definitions is not historically accurate and, pedagogically, does not allow students the room to grow their understanding.

This being said, the courses in the calculus sequence are not history of mathematics courses, and care should be taken to not stray too far afield. There is a way to utilize the historical development of the subject to provide a narrative for the subject, while still focusing on the content required in the course. For this narrative to be both historically accurate and of interest to the general student, it must first focus on solving specific problems first and then eventually presenting the theory to address anomalies occurring when these techniques are pushed too far. This can be done in such a way as to "get students to the same spot" by the end of the course as a traditional sequence would.

Indefinite Integrals Before Definite Integrals

The typical calculus textbook introduces the subject of integration with a brief talk of accumulation functions and areas under curves. Relatively early in such a book, a formal definition of a definite integral is provided. The following is from Stewart.

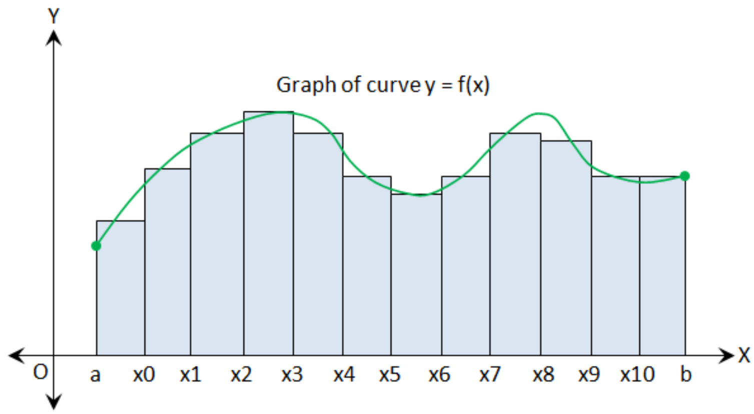
> **Definition.** If f is a function $a \leq x \leq b$, we divide the interval $[a, b]$ into n subintervals of equal width $\Delta x = (b - a)/n$. We let

$x_0(= a),\ x_1,\ x_2,\ \dots,\ x_n(= b)$ be the endpoints of these subintervals and we let $x_1^*,\ x_2^*,\ \dots,\ x_n^*$ be any **sample points** in these subintervals, so x_i^* lies in the ith subinterval $[x_{i-1},\ x_i]$. Then the **definite integral of f from a to b** is

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x$$

Provided that this limit exists and gives the same value for all possible choices of sample points. If it does exist, we say that f is **integrable** on $[a, b]$.

Stewart goes further to provide a picture



and identifies the sum involved as a **Riemann Sum**. Curiously, what is not mentioned is that the integral symbol $\int y\,dx$ was introduced by Leibniz in 1675, 151 years before Riemann's birth (1826). Furthermore, Riemann provided this definition as an extension of Cauchy's definition utilizing left-hand sums in 1823 [Ruch, 2017]. Both Cauchy and Riemann were not interested in computation of integrals, per se, but rather were determining conditions upon which the Fourier series of a function would converge to that function. So, an interesting question to ask is why we are introducing this concept to beginning calculus students with a rigorous definition designed to look at questions of integrability and its connection to trigonometric series?

The idea of computing areas and volumes by summing infinitesimal quantities certainly predates this rigorous definition and, if fact, predates calculus. Much of calculus would not exist without these ideas. This is premise of Bressoud and Apostol, as they start with Archimedes *Method [of Equilibrium]*. This is where history should take a back seat to pedagogy in the calculus sequence. As interesting and important as these results are, students who have just completed the first semester in differential calculus are accustom to differentiation. In this case, it might be prudent to consider the way of thinking in the period between the invention of calculus and the considerations of integrability.

During this time-period, integration was the inverse operation of differentiation. It was well understood at the time that this could be utilized to find areas and volumes, but it was really the inverse operation of differentiation. At this point, calculus students are prepared for this natural progression.

This presentation proposes that the topics in an integral calculus course (typically the second semester of calculus) can be rearranged to look at applications first and then come back to the theory later in the semester, when students have more appreciation of how calculus is utilized as a tool to solve problems. This is consistent with the idea that almost all mathematics was created to solve a specific problem. Foundational aspects are among the last to be developed. With this in mind, the overall sequence can be as follows:

i. Modeling with differential equations and utilizing techniques of integration to solve these equations,
ii. Connecting areas to integration and Leibniz' proof of the Fundamental Theorem of Calculus,

iii. Applications of definite integrals by integrating (summing) infinitesimal quantities (differentials) -No Riemann sums!,
iv. Applications of improper integrals,
v. Approximation of definite integrals by Riemann sums and the definition of the definite integral,
vi. Manipulation and applications of power series as "infinite polynomials,"
vii. Numerical series and questions of convergence.

Of course, this is a general overview. Even though the topics are rearranged, by the end of the semester, all the standard topics will have been covered. The advantage of this approach is that it is more historically accurate and is driven by developing tools to solve problems. Furthermore, being problem centered, it provides grounding for students. Even the theory is driven by questions of approximation and foundational matters.

As a further supplement, we are willing to provide a (very rough) partial first draft of an integral calculus book utilizing this approach. This draft has a number of *Problems in Context* designed to motivate and reflect upon topics and techniques learned. These problems not only can be utilized for assignments, but can also be used as discussion topics in a classroom setting. Email robert.rogers@fredonia.edu for a pdf of this draft.

This work is not complete, but contains sufficient material to support you if you choose to utilize such an approach. You are welcome to use the materials as you can in your teaching. You can contact us with any questions/comments.

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