Some new Hilbert type inequalities and applications

Gao Mingzhe

ABSTRACT. In this paper it is shown that some new Hilbert type integral inequalities can be established by introducing a proper logarithm function. And the constant factor is proved to be the best possible. In particular, for case , the classical Hilbert inequality and its equivalent form are obtained. As applications, some new inequalities which they are equivalent each other are built.

1. INTRODUCTION

Let \( f(x), g(x) \in L^2(0, +\infty) \). Then

\[
\int_0^\infty \int_0^\infty \frac{f(x)g(x)}{x+y} dxdy \leq \pi \left\{ \int_0^\infty f^2(x) \, dx \right\}^{\frac{1}{2}} \left\{ \int_0^\infty g^2(x) \, dx \right\}^{\frac{1}{2}}
\]

(1.1)

This is the famous Hilbert integral inequality, where the coefficient \( \pi \) is the best possible.

In the papers [1-2], the following inequality of the form

\[
\int_0^\infty \int_0^\infty \frac{\ln \frac{x}{y} f(x)g(y)}{x-y} dxdy \leq \pi^2 \left\{ \int_0^\infty f^2(x) \, dx \right\}^{\frac{1}{2}} \left\{ \int_0^\infty g^2(x) \, dx \right\}^{\frac{1}{2}}
\]

(1.2)

was established, and the coefficient \( \pi^2 \) is also the best possible.

Owing to the importance of the Hilbert inequality and the Hilbert type inequality in analysis and applications, some mathematicians have been studying them. Recently, various improvements and extensions of (1.1) and (1.2) appear in a great deal of papers (see [3]-[8] etc.).

The aim of the present paper is to build some new Hilbert type integral inequalities by introducing a proper integral kernel function and by using the
technique of analysis, and to discuss the constant factor of which is related to the Euler number, and then to study some equivalent forms of them. In the sake of convenience, we introduce some notations and define some functions.

Let $0 < \alpha < 1$ and $n$ be a positive integer. Define a function $\zeta^*$ by

$$
\zeta^* (n, \alpha) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(\alpha + k)^n}.
$$

(1.3)

And further define the function $\zeta_2$ by

$$
\zeta_2 = (2n)! \left\{ 2 \zeta^* \left( 2n + 1, \frac{1}{2} \right) \right\}, (n \in N_0)
$$

(1.4)

In order to prove our main results, we need the following lemmas.

**Lemma 1.1.** Let $0 < \alpha < 1$ and $n$ be a nonnegative integer. Then

$$
\int_0^1 t^{\alpha-1} \left( \ln \frac{1}{t} \right)^n \frac{1}{1+t} dt = n! \zeta^* (n + 1, \alpha).
$$

(1.5)

where $\zeta^*$ is defined by (1.3).

This result has been given in the paper [9]. Hence its proof is omitted here.

**Lemma 1.2.** With the assumptions as Lemma 1.1, then

$$
\int_0^\infty u^{\alpha-1} \left( \ln \frac{1}{u} \right)^{2n} \frac{1}{1+u} du = (2n)! \left\{ \zeta^* (2n + 1, \alpha) + \zeta^* (2n + 1, 1 - \alpha) \right\}
$$

(1.6)

where $\zeta^*$ is defined by (1.3).

**Proof.** It is easy to deduce that

$$
\int_0^\infty u^{\alpha-1} \left( \ln \frac{1}{u} \right)^{2n} \frac{1}{1+u} du = \int_0^1 u^{\alpha-1} \left( \ln \frac{1}{u} \right)^{2n} \frac{1}{1+u} du + \\
+ \int_1^\infty u^{\alpha-1} \left( \ln \frac{1}{u} \right)^{2n} \frac{1}{1+u} du = \int_0^1 u^{\alpha-1} \left( \ln \frac{1}{u} \right)^{2n} \frac{1}{1+u} du + \\
+ \int_1^\infty u^{\alpha-1} \left( \ln \frac{1}{u} \right)^{2n} \frac{1}{1+u} du
$$
+ \int_0^1 v^{-\alpha} (\ln v)^{2n} \frac{1}{1+v} dv = \int_0^1 u^{\alpha-1} \left( \ln \frac{1}{u} \right)^{2n} \frac{1}{1+u} du + \\
+ \int_0^1 v^{(1-\alpha)-1} \left( \ln \frac{1}{v} \right)^{2n} \frac{1}{1+v} dv.

By using Lemma 1.1, the equality (1.6) is obtained at once.
Throughout the paper, we define \( (\ln \frac{x}{y})^0 = 1 \), when \( x = y \).

2. MAIN RESULTS

We are ready now to formulate our main results.

**Theorem 2.1.** Let \( f \) and \( g \) be two real functions, and \( n \) be a nonnegative integer, If

\[
\int_0^\infty f^2(x) \, dx < +\infty \quad \text{and} \quad \int_0^\infty g^2(x) \, dx < +\infty,
\]

then

\[
\int_0^\infty \int_0^\infty \left( \ln \frac{x}{y} \right)^{2n} f(x) \, g(y) \frac{1}{x+y} \, dx \, dy \leq \\
\leq \left( \pi^{2n+1} E_n \right) \left\{ \int_0^\infty f^2(x) \, dx \right\} \frac{1}{2} \left\{ \int_0^\infty g^2(x) \, dx \right\} \frac{1}{2}, \tag{2.1}
\]

where the constant factor \( \pi^{2n+1} E_n \) is the best possible, and that \( E_0 = 1 \) and \( E_n \) is the Euler number, viz. \( E_1 = 1, E_2 = 5, E_3 = 61, E_4 = 1385, E_5 = 50521, \) etc.

**Proof.** We may apply the Cauchy inequality to estimate the left-hand side of (2.1) as follows:

\[
\int_0^\infty \int_0^\infty \left( \ln \frac{x}{y} \right)^{2n} f(x) \, g(y) \frac{1}{x+y} \, dx \, dy = 
\]
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$$= \int_{0}^{\infty} \int_{0}^{\infty} \left( \frac{\ln \frac{x}{y}}{x+y} \right)^{2n} \left( \frac{x}{y} \right) \frac{1}{2} f(x) \left( \frac{\ln \frac{x}{y}}{x+y} \right)^{2n} \left( \frac{y}{x} \right)^{1/4} g(y) dx dy \leq$$

$$\leq \left\{ \int_{0}^{\infty} \int_{0}^{\infty} \left( \frac{\ln \frac{x}{y}}{x+y} \right)^{2n} \left( \frac{x}{y} \right)^{1/2} f^2(x) \right\}^{1/2} \left\{ \int_{0}^{\infty} \int_{0}^{\infty} \left( \frac{\ln \frac{x}{y}}{x+y} \right)^{2n} \left( \frac{x}{y} \right)^{1/2} g^2(x) \right\}^{1/2} =$$

$$= \left( \int_{0}^{\infty} \omega(x) f^2(x) dx \right)^{1/2} \left( \int_{0}^{\infty} \omega(x) g^2(x) dx \right)^{1/2} \quad (2.2)$$

where $\omega(x) = \int_{0}^{\infty} \left( \frac{\ln \frac{x}{y}}{x+y} \right)^{2n} \left( \frac{x}{y} \right)^{1/2} dy$, By using Lemma 1.2, it is easy to deduce that

$$\omega(x) = \int_{0}^{\infty} \left( \frac{\ln \frac{x}{y}}{x+y} \right)^{2n} \left( \frac{x}{y} \right)^{1/2} dy = \int_{0}^{u} \left( \frac{1}{\ln \frac{u}{1+u}} \right)^{2n} \frac{1}{1+u} du = \zeta_2. \quad (2.3)$$

where $\zeta_2$ is defined by (1.4). Based on (1.3) and (1.4), we have

$$\zeta_2 = (2n)! \left\{ 2\zeta^* \left( 2n + 1, \frac{1}{2} \right) \right\} = (2n)! \sum_{k=0}^{\infty} \frac{(-1)^k}{\left( \frac{1}{2} + k \right)^{2n+1}} =$$

$$= (2n)! \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^{2n+1}}. \quad (2.4)$$

It is known from the paper [10] that

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{(\frac{1}{2} + k)^{2n+1}} = \frac{\pi^{2n+1}}{2^{2n+2} (2n)!} E_n. \quad (2.4)$$

where $E_n$ is the Euler number, viz. $E_1 = 1$, $E_2 = 5$, $E_3 = 61$, $E_4 = 1385$, $E_5 = 50521$, etc.

Since $\sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} = \frac{\pi}{4}$, we can define $E_0 = 1$. As thus, the relation (2.4) is also valid when $n = 0$. So, we get from (2.3) and (2.4) that
\[ \omega (x) = \pi^{2n+1} E_n, \quad (2.5) \]

It follows from (2.2) and (2.5) that the inequality (2.1) is valid.

It remains to need only to show that \( \pi^{2n+1} E_n \) in (2.1) is the best possible.

\( \forall \varepsilon > 0 \). Define two functions by
\[ \tilde{f}(x) = \begin{cases} 0 & \text{if } x \in (0, 1) \\ x^{-\frac{1+\varepsilon}{2}} & \text{if } x \in [1, \infty) \end{cases} \]
and
\[ \tilde{g}(y) = \begin{cases} 0 & \text{if } y \in (0, 1) \\ y^{-\frac{1+\varepsilon}{2}} & \text{if } y \in [1, \infty) \end{cases}. \]

It is easy to deduce that
\[ \int_0^{+\infty} \tilde{f}^2(x) \, dx = \int_0^{+\infty} \tilde{g}^2(y) \, dy = \frac{1}{\varepsilon}. \]

If \( \pi^{2n+1} E_n \) is not the best possible, then there exists \( C > 0 \), such that
\[ C < \pi^{2n+1} E_n \]
and
\[ S(\tilde{f}, \tilde{g}) = \int_0^{+\infty} \int_0^{+\infty} \frac{(\ln \frac{z}{y})^{2n} \tilde{f}(x) \tilde{g}(y)}{x+y} \, dx \, dy \]
\[ \leq C \left( \int_0^{+\infty} \tilde{f}^2(x) \, dx \right)^{\frac{1}{2}} \left( \int_0^{+\infty} \tilde{g}^2(y) \, dy \right)^{\frac{1}{2}} = \frac{C}{\varepsilon}. \quad (2.6) \]

On the other hand, we have
\[ S(\tilde{f}, \tilde{g}) = \int_0^{+\infty} \int_0^{+\infty} \left\{ x^{-\frac{1+\varepsilon}{2}} \right\} \left\{ (\ln \frac{z}{y})^{2n} y^{-\frac{1+\varepsilon}{2}} \right\} \, dx \, dy = \]
\[ = \int_0^{+\infty} \int_0^{+\infty} \left\{ \frac{(\ln \frac{z}{y})^{2n} y^{-\frac{1+\varepsilon}{2}}}{x+y} \, dy \right\} \left\{ x^{-\frac{1+\varepsilon}{2}} \right\} \, dx = \]
\[ = \int_0^{+\infty} \int_0^{+\infty} \left\{ \frac{(\ln \frac{1}{y})^{2n} u^{-\frac{1+\varepsilon}{2}}}{1+u} \, du \right\} \left\{ x^{-1-\varepsilon} \right\} \, dx = \]
\[ = \frac{1}{\varepsilon} \int_0^{+\infty} u^{-\frac{1+\varepsilon}{2}} \left( \ln \frac{1}{u} \right)^{2n} \frac{1}{1+u} \, du. \quad (2.7) \]
When $\varepsilon$ is small enough, based on (2.3) and (2.5) we can write the integral of (2.7) in the following form:

$$\int_0^\infty u^{-\frac{1+\varepsilon}{2}} \left(\ln \frac{1}{u}\right)^{2n} \frac{1}{1+u} du = \pi^{2n+1} E_n + o(1). \ (\varepsilon \to 0) \quad (2.8)$$

It follows from (2.7) and (2.8) that

$$S(\tilde{f}, \tilde{g}) = \frac{1}{\varepsilon} \left\{ (\pi^{2n+1} E_n) + o(1) \right\}, \ (\varepsilon \to 0) \quad (2.9)$$

When $\varepsilon$ is small enough, it is obvious that the inequality (2.6) is in contradiction with (2.9). Therefore, the constant factor $\pi^{2n+1} E_n$ in (2.1) is the best possible. Thus the proof of Theorem is completed.

In particular, when $n = 0$, the inequality (2.1) is reduced to (1.1). Thereby the inequality (2.1) is an extension of (1.1).

Notice that the constant factor $\pi^{2n+1} E_n$ in (2.1) can be reduced to $\pi^3$, if $n = 1$. Hence we have the following important result.

**Corollary 2.2.** With the assumptions as Theorem 2.1, then

$$\int_0^\infty \int_0^\infty \left(\ln \frac{x}{y}\right)^2 f(x) g(y) \frac{dx dy}{x+y} \leq \pi^3 \left\{ \int_0^\infty f^2(x) \, dx \right\}^{\frac{1}{2}} \left\{ \int_0^\infty g^2(x) \, dx \right\}^{\frac{1}{2}} \quad (2.10)$$

where the constant factor $\pi^2$ is the best possible.

**Corollary 2.3.** Let $f(x)$ be a real functions, and $n$ be a nonnegative integer, If $\int_0^\infty f^2(x) \, dx < +\infty$, then

$$\int_0^\infty \int_0^\infty \left(\ln \frac{x}{y}\right)^2 f(x) g(y) \frac{dx dy}{x+y} \leq (\pi^{2n+1} E_n) \int_0^\infty f^2(x) \, dx, \quad (2.11)$$

where the constant factor $\pi^{2n+1} E_n$ is the best possible, and that $E_0 = 1$ and $E_n$ is the Euler number, viz. $E_1 = 1$, $E_2 = 5$, $E_3 = 61$, $E_4 = 1385$, $E_5 = 50521$, etc.

**Corollary 2.4.** Let $f(x)$ be a real functions, and $n$ be a nonnegative integer, If $\int_0^\infty f^2(x) \, dx < +\infty$, then
\[ \int_0^\infty \int_0^\infty \frac{\left( \ln \frac{y}{x} \right)^2 f(x) g(y)}{x+y} \, dx \, dy \leq \pi^3 \int_0^\infty f^2(x) \, dx \] (2.12)

where the constant factor \( \pi^3 \) is the best possible.

Notice that \( E_0 = 1 \), so we obtain (1.1) from (2.1) immediately when \( n = 0 \). Thereby the inequality (2.1) is an extension of (1.1).

3. SOME APPLICATIONS

As applications, we will build the following inequalities.

**Theorem 3.1.** Let \( n \) be a nonnegative integer. If \( \int_0^\infty f^2(x) \, dx < +\infty \), then

\[ \int_0^\infty \left\{ \int_0^\infty \frac{\left( \ln \frac{y}{x} \right)^{2n}}{x+y} f(x) \, dx \right\}^2 \, dy \leq \left( \pi^{2n+1} E_n \right)^2 \int_0^\infty f^2(x) \, dx, \] (3.1)

where \( \left( \pi^{2n+1} E_n \right)^2 \) in (3.1) is the best possible, and that \( E_0 = 1 \) and \( E_n \) is the Euler number, viz. \( E_1 = 1, E_2 = 5, E_3 = 61, E_4 = 1385, E_5 = 50521, \) etc. And the inequality (3.1) is equivalent to (2.1).

**Proof.** Assume that the inequality (2.1) is valid. Setting a real function \( g(y) \) as

\[ g(y) = \int_{0}^{\infty} \frac{\left( \ln \frac{y}{x} \right)^{2n}}{x+y} f(x) \, dx, \quad y \in (0, +\infty) \]

By using (2.1), we have

\[ \int_0^\infty \left\{ \int_0^\infty \frac{\left( \ln \frac{y}{x} \right)^{2n}}{x+y} f(x) \, dx \right\}^2 \, dy = \int_0^\infty \int_0^\infty \frac{\left( \ln \frac{y}{x} \right)^{2n}}{x+y} f(x) \, g(y) \, dx \, dy \leq \]

\[ \leq \left( \pi^{2n+1} E_n \right) \left\{ \int_0^\infty f^2(x) \, dx \right\} \frac{1}{2} \left\{ \int_0^\infty g^2(y) \, dy \right\} \frac{1}{2} = \]
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\[
= (\pi^{2n+1} E_n) \left\{ \int_0^\infty f^2(x) \, dx \right\}^{\frac{1}{2}} \left\{ \int_0^\infty \left( \int_0^\infty \frac{(\ln \frac{x}{y})^{2n}}{x+y} f(x) \, dx \right)^2 \, dy \right\}^{\frac{1}{2}}
\]  

(3.2)

where \( E_0 = 1 \) and \( E_n \) is the Euler number, viz. \( E_1 = 1, E_2 = 5, E_3 = 61, E_4 = 1385, E_5 = 50521, \) etc.

It follows from (3.2) that the inequality (3.1) is valid after some simplifications.

On the other hand, assume that the inequality (3.1) keeps valid, by applying in turn Cauchy’s inequality and (3.1), we have

\[
\int_0^\infty \int_0^\infty \frac{(\ln \frac{x}{y})^{2n}}{x+y} f(x) g(y) \, dx \, dy \leq \\
\leq \left\{ \int_0^\infty \left( \int_0^\infty \frac{(\ln \frac{x}{y})^{2n}}{x+y} f(x) \, dx \right)^2 \, dy \right\}^{\frac{1}{2}} \left\{ \int_0^\infty g^2(y) \, dy \right\}^{\frac{1}{2}} \\
\leq \left( \pi^{2n+1} E_n \right)^2 \left\{ \int_0^\infty f^2(x) \, dx \right\}^{\frac{1}{2}} \left\{ \int_0^\infty g^2(y) \, dy \right\}^{\frac{1}{2}} = \\
= \left( \pi^{2n+1} E_n \right)^2 \left\{ \int_0^\infty f^2(x) \, dx \right\}^{\frac{1}{2}} \left\{ \int_0^\infty g^2(y) \, dy \right\}^{\frac{1}{2}}
\]  

(3.3)

where \( E_0 = 1 \) and \( E_n \) is the Euler number, viz. \( E_1 = 1, E_2 = 5, E_3 = 61, E_4 = 1385, E_5 = 50521, \) etc.

If the constant factor \( \left( \pi^{2n+1} E_n \right)^2 \) in (3.1) is not the best possible, then it is known from (3.3) that the constant factor \( \pi^{2n+1} E_n \) in (2.1) is also not the best possible. This is a contradiction. Theorem is proved.

**Corollary 3.2.** With the assumptions as Theorem 3.1, then

\[
\int_0^\infty \left\{ \int_0^\infty \frac{(\ln \frac{x}{y})^{2n}}{x+y} f(x) \, dx \right\}^2 \, dy \leq \pi^6 \int_0^\infty f^2(x) \, dx,
\]  

(3.4)
where the constant factor $\pi^6$ is the best possible. Inequality (3.4) is equivalent to (2.10).
In particular, for case $n=0$, based on Theorem 3.1 we have the following result.

**Corollary 3.3.** If $\int_0^\infty \int_0^\infty \frac{1}{x+y} f(x) \, dx \, dy < +\infty$, then
\[
\int_0^\infty \left\{ \int_0^\infty \frac{1}{x+y} f(x) \, dx \right\}^2 \leq \pi^2 \int_0^\infty f^2(x) \, dx.
\] (3.5)

where $\pi^2$ in (3.5) is the best possible, And the inequality (3.5) is equivalent to (1.1).
The proofs of Corollaries 3.2 and 3.3 are similar to one of Theorem 3.1, it is omitted here.

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Department of Mathematics and Computer Science,
Normal College of Jishou University,
Hunan Jishou, 416000, Peoples Republic China
E-mail: mingzhegao@163.com