On Hardy-type integral inequalities involving many functions

O.O. Fabelurin, A. G. Adeagbo-Sheikh

In memory of Professor C. O. Imoru

ABSTRACT. In this paper, we use Jensen’s inequality, and a modification of an inequality involving some constants, to obtain a Hardy-type integral inequality involving many functions. Our inequality features a refinement term and is sharper than the inequality of Cheung, Hanjiš and Pečarić (2000) in the segment (1, ∞) of the real line.

1. INTRODUCTION AND PRELIMINARIES

The classical Hardy’s inequality (1920) states that, for any $p > 1$ and any integrable function $f(x) \geq 0$ on $(0, \infty)$, if $F(x) = \int_0^x f(x)dx$, then

$$\int_0^\infty \left[ \frac{F(x)}{x} \right]^p dx < \left[ \frac{p}{p-1} \right]^p \int_0^\infty f(x)^p dx$$  (1.1)

Unless $f \equiv 0$, where the constant here is best possible.

In view of the usefulness of the inequality 1.1 in analysis and its applications, it has received considerable attention and a number of papers have appeared in [1-12], which deal with its various improvements, extensions, generalizations and applications.

Of particular importance, relevance and great motivation for this research, is the following work of Cheung, Hanjiš and Pečarić [6][Theorem 1]:

For any $i = 1, \ldots, n$, let $f_i : (0, \infty) \to (0, \infty)$ be absolutely continuous, let $g_i : (0, \infty) \to (0, \infty)$ be integrable, and $p_i > q_i > 0, m_i > q_i$ be real numbers such that $\sum q_i = 1$ and

$$1 + \left( \frac{p_i}{m_i - q_i} \right) x f_i'(x) \geq \frac{1}{\gamma_i} \ a.e$$

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for some constants $\gamma_i > 0$. If we denote $p = \sum p_i$, $m = \sum m_i$ and

$$\eta_i(x) = \frac{1}{f_i(x)} \int_0^x \frac{f_i(t)g_i(t)}{t} dt \quad x \in (0, \infty),$$

then

$$\int_0^\infty x^{-m} \prod_i \eta_i^{p_i}(x) \, dx \leq \left( \prod_j C_j^{-p_j} \right) \sum_i q_i C_i^{p_i/q_i} \left[ \frac{p_i \gamma_i}{m_i - q_i} \right] \int_0^\infty x^{-(m_i/q_i)} g_i^{p_i/q_i}(x) \, dx.$$  \hfill (1.2)

In establishing their result, Cheung et al made use of Holder’s inequality. In this work, our main tool is the inequality of Jensen for convex functions.

2. MAIN RESULTS

For our main result we shall need the following lemma

**Lemma 2.1.** Let $p \geq q > 0$ and $r \neq 1$ be real numbers. Let $f : [a, b] \to (0, \infty)$ be absolutely continuous and let $g : [a, b] \to [0, \infty)$ be integrable with $0 < a \leq b < \infty$. Let

$$\varphi_a(x) = \int_a^x \frac{f(t)g(t)}{f(x)} t \, dt, \varphi_b(x) = \int_x^b \frac{f(t)g(t)}{f(x)} t \, dt, \delta = \frac{q}{p} (1 - r)$$

for $r \neq 1,$

$$1 + \left( \frac{p}{q} \right)^2 \left( \frac{1}{r-1} \right) \frac{xf'(x)}{f(x)} \geq \frac{1}{\lambda} > 0 \quad \text{a.e for } r > 1 \quad (2.1)$$

and

$$1 - \left( \frac{p}{q} \right)^2 \left( \frac{1}{1-r} \right) \frac{xf'(x)}{f(x)} \geq \frac{1}{\lambda} > 0 \quad \text{a.e for } r > 1 \quad (2.2)$$

for some constant $\lambda > 0$.

Then if $r > 1$

$$\int_a^b x^{-r} \varphi_a^{p/q}(x) \, dx + \left( \frac{p}{q} \right) \frac{\lambda}{r-1} b^{1-r} \varphi_a^{p/q}(b) \leq \ldots$$
\[
\leq \lambda \left( \left( \frac{p}{q} \right) \frac{1}{1 - \frac{r}{q}} \right)^{p/q} \int_a^b x^{-r} g^{p/q}(x) dx
\quad (2.3)
\]
and for \( r < 1 \)

\[
\int_a^b x^{-r} \varphi_b^{p/q}(x) dx + \left( \frac{p}{q} \right) \frac{\lambda}{1 - r} a^{1-r} \varphi_b^{p/q}(a) \leq \lambda \left( \left( \frac{p}{q} \right) \frac{1}{1 - \frac{r}{q}} \right)^{p/q} \int_a^b x^{-r} g^{p/q}(x) dx
\quad (2.4)
\]

\textbf{Proof.} The following adaptations of Jensen’s inequality for convex functions \([1-4]\) will be used in the proof of the Lemma 2.1

\[
\left[ \int_a^x d\lambda(t) \right]^{1-\tau} \left[ \int_a^x h(x,t) \frac{1}{\tau} d\lambda(t) \right]^{\tau} \leq \int_a^x h(x,t) d\lambda(t)
\quad (2.5)
\]

\[
\left[ \int_a^b d\lambda(t) \right]^{1-\tau} \left[ \int_a^b h(x,t) \frac{1}{\tau} d\lambda(t) \right]^{\tau} \leq \int_a^b h(x,t) d\lambda(t)
\quad (2.6)
\]
where \( h(x,t) \geq 0 \) for \( x \geq 0, t \geq 0, \lambda \) is non-decreasing and \( \tau \geq 1 \).

Let \( h(x,t) = x^\delta t^{r(1+\delta)} \left( \frac{f(x)g(t)}{f(x)g(t)} \right)^\tau dt \) and \( \delta = \frac{1-r}{\tau} \).

Using the above definitions of \( h, \lambda \) and \( \delta \) in 2.5 and 2.6, we obtain

\[
\left[ -\delta^{-1} \right]^{1-\tau} \left[ x^{-\delta} - a^{-\delta} \right]^{1-\tau} x^\delta \varphi_a^{\tau}(x) \leq x^\delta \theta_a(x) \quad \forall x \in [a,b].
\quad (2.7)
\]

\[
\left[ \delta^{-1} \right]^{1-\tau} \left[ x^{-\delta} - b^{-\delta} \right]^{1-\tau} x^\delta \varphi_b^{\tau}(x) \leq x^\delta \theta_b(x) \quad \forall x \in [a,b].
\quad (2.8)
\]

Where \( \theta_a(x) = \int_a^x t^{(r-1)(1+\delta)} \left[ \frac{f(t)g(t)}{f(x)g(t)} \right]^\tau dt \), \( \theta_b(x) = \int_x^b t^{(r-1)(1+\delta)} \left[ \frac{f(t)g(t)}{f(x)g(t)} \right]^\tau dt \).

Multiply through 2.7 and 2.8 by \( x^{-1} \) and then integrate with respect to \( x \) on \( [a, b] \) to get

\[
\left[ -\delta^{-1} \right]^{1-\tau} \int_a^b \left[ x^{-\delta} - a^{-\delta} \right]^{1-\tau} x^{\delta-1} \varphi_a^{\tau}(x) dx \leq \int_a^b x^{\delta-1} \theta_a(x) dx
\quad (2.9)
\]

\[
\left[ \delta^{-1} \right]^{1-\tau} \int_a^b \left[ x^{-\delta} - b^{-\delta} \right]^{1-\tau} x^{\delta-1} \varphi_b^{\tau}(x) dx \leq \int_a^b x^{\delta-1} \theta_b(x) dx.
\quad (2.10)
\]
Integrate the RHS of 2.9 by parts and then factorise to obtain
\[ \int_a^b x^{-1} \theta_a(x) \left( 1 + \frac{\tau^2}{r-1} \frac{x f'(x)}{f(x)} \right) dx = \delta^{-1} b \delta \theta_a(b) - \delta^{-1} \int_a^b f^\tau(x) x^{-\delta-1} dx \]

Suppose that for some constant \( \lambda > 0 \),
\[ \left( 1 + \frac{\tau^2}{r-1} \frac{x f'(x)}{f(x)} \right) > \frac{1}{\lambda} \quad \text{a.e.} \quad (2.11) \]
then
\[ \int_a^b x^{-1} \theta_a(x) \left( \frac{1}{\lambda} \right) dx \leq \int_a^b x^{-1} \theta_a(x) \left( 1 + \frac{\tau^2}{r-1} \frac{x f'(x)}{f(x)} \right) dx = \delta^{-1} b \delta \theta_a(b) - \delta^{-1} \int_a^b f^\tau(x) x^{-\delta-1} dx. \quad (2.12) \]

Whence on arranging
\[ \int_a^b x^{-1} \theta_a(x) dx + \lambda \left( -\delta^{-1} \right) b \delta \theta_a(b) \leq \lambda \left( -\delta^{-1} \right) \int_a^b f^\tau(x) x^{-\delta-1} dx. \quad (2.13) \]
Now it follows from 2.7 and the fact that \( \lambda \left[ -\delta^{-1} \right] > 0 \) for \( r > 1 \) that
\[ \lambda \left( -\delta^{-1} \right)^{2-\tau} \left[ b^{-\delta} - a^{-\delta} \right]^{1-\tau} b \delta \varphi^\tau_a(b) \leq \lambda \left( -\delta^{-1} \right) b \delta \theta_a(b), \quad \text{for } b \in [a, b]. \quad (2.14) \]
Combining 2.13, 2.14 and 2.9 yields
\[ \left[ -\delta^{-1} \right]^{1-\tau} \int_a^b \left[ x^{-\delta} - a^{-\delta} \right]^{1-\tau} x^{-\delta-1} \varphi^\tau_a(x) dx + \lambda \left( -\delta^{-1} \right)^{2-\tau} \left[ b^{-\delta} - a^{-\delta} \right]^{1-\tau} b \delta \varphi^\tau_a(b) \leq \lambda \left( -\delta^{-1} \right) \int_a^b f^\tau(x) x^{-\delta-1} dx. \quad (2.15) \]
Use the fact that \( \left[ x^{-\delta} - a^{-\delta} \right]^{1-\tau} \geq \left[ x^{-\delta} \right]^{1-\tau} \quad \forall x \in [a, b] \quad \text{and} \quad \left[ -\delta^{-1} \right]^{1-\tau} > 0 \quad \text{for} \quad \tau \geq 1 \quad \text{and} \quad r > 1 \) to reduce 2.15 to
\[ \int_a^b x^{\delta-1} \varphi^\tau_a(x) dx + \lambda \left( -\delta^{-1} \right) b \delta \varphi^\tau_a(b) \leq \lambda \left( -\delta^{-1} \right)^\tau \int_a^b f^\tau(x) x^{\delta-1} dx. \quad (2.16) \]
Similarly, for the case $r < 1$, start with inequality 2.10 and follow the same arguments with slight modifications to the conditions. Specifically, use

$$\left(1 - \frac{\tau^2}{1 - r} \frac{xf'(x)}{f(x)}\right) > \frac{1}{\lambda} \quad \text{a.e. for some constant } \lambda > 0,$$

with $[\delta^{-1}] > 0$, $[x^{-\delta} - b^{-\delta}]^{1-\tau} \geq [x^{-\delta}]^{1-\tau} \forall x \in [a, b]$ to yield

$$\int_a^b x^{\delta-1} x^\delta \varphi(x) dx + \lambda (\delta^{-1}) a^\delta \varphi(a) \leq \lambda (\delta^{-1}) \int_a^b f^\tau(x)x^{-\delta} dx. \quad (2.18)$$

Finally, observe that if $p \geq q > 0$, then $p/q \geq 1$. Thus, Inequalities 2.1, 2.2, 2.3 and 2.4 follow immediately from 2.11, 2.17, 2.16 and 2.18 respectively by recalling that $\delta = (1 - r)/\tau$ and letting $\tau = \frac{p}{q}$.

The following Theorem is an improvement over the result of Cheung, Hanjš and Pečarić[6,Theorems 1 and 2].

**Theorem 2.1.** For any $i = 1, ..., n$, let $f_i : [a, b] \rightarrow (0, \infty)$ be absolutely continuous, let $g_i : [a, b] \rightarrow [0, \infty)$ be integrable with $0 < a \leq b < \infty$. Let $p_i \geq q_i > 0$, $m_i \neq q_i$ be real numbers such that $\sum q_i = 1$,

$$1 + \left(\frac{p_i^2}{q_i(m_i - q_i)}\right) \frac{x^\tau f'(x)}{f_i(x)} \geq \frac{1}{\lambda_i} > 0 \quad \text{a.e} \quad (2.19)$$

and

$$1 - \left(\frac{p_i^2}{q_i(m_i - q_i)}\right) \frac{fu(x)}{f_i(x)} \geq \frac{1}{\lambda_i} > 0 \quad \text{a.e} \quad (2.20)$$

for some constant $\lambda_i > 0$. If we denote $p = \sum p_i$, $m = \sum m_i$, and

$$\varphi_{a,i}(x) = \int_a^x f_i(t)g_i(t) dt, \varphi_{b,i}(x) = \int_x^b f_i(t)g_i(t) dt, \quad x \in (0, \infty),$$

then for $m_i > q_i$

$$\int_a^b \prod_{i=1}^n \left[x^{-m_i} r_{a,i}^{p_i}(x)\right] dx + \prod_{i=1}^n C_j^{-p_j} \sum_{i=1}^n q_i C_i^{p_i/q_i} \frac{\lambda p_i}{m_i - q_i} h^{1-m_i/q_i} \varphi_{a,i}(b) \leq$$
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\[
\leq \prod_{j=1}^{n} C_j^{-p_j} \sum_{i=1}^{n} q_i C_i^{p_i/q_i} \lambda_i \left( \frac{p_i}{m_i - q_i} \right)^{p_i/q_i} \int_0^\infty x^{-m_i/q_i} g_i(x)^{p_i/q_i} dx \tag{2.21}
\]

and for \( m_i < q_i \)

\[
\int_a^b \prod_{i=1}^{n} \left[ x^{-m_i/q_i} \varphi_{a,i}(x) \right] dx + \prod_{j=1}^{n} C_j^{-p_j} \sum_{i=1}^{n} q_i C_i^{p_i/q_i} \frac{\lambda_i p_i}{q_i - m_i} b^{1-m_i/q_i} \varphi_{b,i}^{p_i/q_i}(a) \leq \prod_{j=1}^{n} C_j^{-p_j} \sum_{i=1}^{n} q_i C_i^{p_i/q_i} \lambda_i \left( \frac{p_i}{q_i - m_i} \right)^{p_i/q_i} \int_0^\infty x^{-m_i/q_i} g_i(x)^{p_i/q_i} dx \tag{2.22}
\]

Proof. Firstly, observe that \( m_i > q_i \) implies that \( m_i/q_i > 1 \). Consequently, it follows from Lemma 2 for the case \( r = m_i/q_i > 1 \) that

\[
\int_a^b x^{-m_i/q_i} \varphi_{a,i}(x) dx + \frac{\lambda_i p_i}{m_i - q_i} b^{1-m_i/q_i} \varphi_{b,i}^{p_i/q_i}(b) \leq \lambda_i \left( \frac{p_i}{m_i - q_i} \right)^{p_i/q_i} \int_a^b x^{-m_i/q_i} g_i(x)^{p_i/q_i} dx \tag{2.23}
\]

Now for any \( C_i > 0 \), we have by the arithmetic-geometric inequality [6], that

\[
\prod_{i=1}^{n} \left[ x^{-m_i/q_i} \varphi_{a,i}(x) \right] = \prod_{i=1}^{n} \left( \left( x^{-(m_i/p_i)} C_i^{p_i/q_i} \varphi_{a,i}(x) \right)^{p_i/q_i} \right)^{q_i} C_i^{-p_i} = \prod_{j=1}^{n} C_j^{-p_j} \prod_{i=1}^{n} \left[ \left( x^{-(m_i/p_i)} C_i^{p_i/q_i} \varphi_{a,i}(x) \right)^{p_i/q_i} \right]^{q_i} \leq \prod_{j=1}^{n} C_j^{-p_j} \sum_{i=1}^{n} q_i C_i^{p_i/q_i} x^{-m_i/q_i} \varphi_{a,i}^{p_i/q_i}(x) \tag{2.24}
\]

Integrate both sides of 2.24 with respect to \( x \) on \([a, b]\), to obtain

\[
\int_a^b \prod_{i=1}^{n} \left[ x^{-m_i/q_i} \varphi_{a,i}(x) \right] dx \leq \prod_{j=1}^{n} C_j^{-p_j} \sum_{i=1}^{n} q_i C_i^{p_i/q_i} \int_a^b x^{-m_i/q_i} \varphi_{a,i}^{p_i/q_i}(x) dx. \tag{2.25}
\]
We combine inequalities 2.25 and 2.23, expanding and rearranging (letting \( m = \sum m_i \)) to obtain

\[
\int_a^b x^{-m} \prod_{i=1}^n \left[ \varphi_{a,i}^{p_i}(x) \right] \, dx + \prod_{j=1}^n C_j^{-p_j} \sum_{i=1}^n q_i C_i^{p_i/q_i} \frac{\lambda_i p_i}{m_i - q_i} b^{1-m_i/q_i} \varphi_{a,j}^{p_i/q_i}(b) \leq \\
\leq \prod_{j=1}^n C_j^{-p_j} \sum_{i=1}^n q_i C_i^{p_i/q_i} \lambda_i \left( \frac{p_i}{m_i - q_i} \right)^{p_i/q_i} \int_0^\infty x^{-m_i/q_i} g_i(x)^{p_i/q_i} \, dx. \quad (2.26)
\]

For the case when \( m_i < q_i \), it follows from Lemma 2.1 by using similar arguments to those in the proof for the case \( m_i > q_i \).

**Remark 2.2.** If we let \( a \to 0^+ \) and \( b \to \infty \) then 2.26 reduces to

\[
\int_0^\infty x^{-m} \prod_{i=1}^n \left[ \varphi_{a,i}^{p_i}(x) \right] \, dx \leq \\
\leq \prod_{j=1}^n C_j^{-p_j} \sum_{i=1}^n q_i C_i^{p_i/q_i} \lambda_i \left( \frac{p_i}{m_i - q_i} \right)^{p_i/q_i} \int_0^\infty x^{-m_i/q_i} g_i(x)^{p_i/q_i} \, dx. \quad (2.27)
\]

We claim that under certain conditions, the above inequality is sharper than the one by Cheung, Hanjš and Pečarić [6, Theorem 1]:

\[
\int_0^\infty x^{-m} \prod_i \left[ \varphi_i^{p_i}(x) \right] \, dx \leq \\
\leq \left( \prod_j C_j^{-p_j} \right) \sum_i q_i C_i^{p_i/q_i} \left[ \frac{p_i \gamma_i}{m_i - q_i} \right] \int_0^\infty x^{-m_i/q_i} g_i^{p_i/q_i}(x) \, dx. \quad (2.28)
\]

Precisely, if \( \gamma_i \in (1, \infty) \)

To justify our claim, we recall the following conditions:

\[
1 + \frac{p_i}{q_i} \left( \frac{p_i}{m_i - q_i} \right) \frac{x f_i'(x)}{f_i(x)} \geq \frac{1}{\lambda_i} > 0 \quad \text{a.e}
\]
and

\[ 1 + \left( \frac{p_i}{m_i - q_i} \right) \frac{xf'(x)}{f_i(x)} \geq \frac{1}{\gamma_i} > 0 \quad \text{a.e.} \]

By comparison, it is obvious that

\[ 1 + \frac{p_i}{q_i} \left( \frac{p_i}{m_i - q_i} \right) \frac{xf'(x)}{f_i(x)} \geq 1 + \left( \frac{p_i}{m_i - q_i} \right) \frac{xf'(x)}{f_i(x)} \]

and \( \frac{1}{\lambda_i} \geq \frac{1}{\gamma_i} > 0. \) Thus

\[ 0 \leq \lambda_i \leq \gamma_i. \]

Finally, we now note that if \( \gamma_i \in (1, \infty) \), then

\[ 0 \leq \lambda_i \leq \gamma_i \leq (\gamma_i)^{p_i/q_i}. \]

for \( p_i/q_i \geq 1. \)

We make similar claim for 2.22.

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Department of Mathematics, Obafemi Awolowo University, Ile-Ife, Osun State, Nigeria.
E-mail: fabelur@yahoo.com

Department of Mathematics, Obafemi Awolowo University, Ile-Ife, Osun State, Nigeria.
E-mail: adesheikh2000@yahoo.co.uk