



Euler and music. A forgotten arithmetic function by Euler

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Dedicated to the 100th Anniversary of the famous musician David Lerner (1909-)

ABSTRACT. We study certain properties of an arithmetic function by Euler, having application in the theory of music.

MAIN RESULTS

1. Since the time of Ancient Greece, mathematicians and non-mathematicians have tried to find connections between mathematic and music. Especially are well-known the findings of Pythagora and his followers on the relations of natural numbers, the lengths of a vibrating string, and the pitches produced by this string.

The Pythagoreans were interested also in the number mysticism and studied these relations by experimenting with a monochored.

They discovered that a string whose lenght is subdivided in a ratio represented by a fraction of two positive integers produces a note that is in harmony with the note produced by the full string: if the ratio is 1:2 then the result is an octave, with 2:3 one gets a perfect fifth, with 3:4 a perfect fourth, etc.

Of particular importance was the discovery of the so-called Pythagorean comma. In all pitch systems that are based on perfect octaves and perfect fifths there is a discrepancy between the interval of seven octaves and the

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interval of twelve fifths, although both have to be considered as equal in musical terms.

In musical practice the Pythagorean comma causes serious problems. So in the past numerous approaches were developed to find tunings for instruments that reduce these problems to a minimum. The tuning that today is known best and used most often in European music is the equal temperament or well temperament tuning. This tuning became popular during the baroque era and most notably by "The well-tempered Clavier", Bach's grand collection of preludes and fugues that impressively demonstrated the possibility of letting all keys sound equally well. Of course one could say also equally bad, since in the equal temperament none of the intervals but the octaves are perfect any more, i.e. the ratio mentioned above are no longer valid.

In the equal temperament every octave is subdivided into twelve half-steps all of which have the same frequency ratio of $2^{1/12}$, where in the terminology above the 2 is to be read as 2:1, i.e., the frequency ratio of an octave. All frequencies of the pitches of the equal tempered twelve-tone scale can be expressed by the geometric sequence

$$f_i = f_0 \cdot 2^{i/12},$$

where f_0 is a fixed frequency, e.g., the standard pitch a' (440 Hz) and i is the half-step distance of the target note from the note with the frequency f_0 . Then, f_i is the frequency of the target note.

In modern times, Leonard Euler (1707-1783) was one of the first who tried to use mathematical methods in order to deal with the consonance/dissonance problem. In his work, too, ratios of natural numbers, reflecting frequency ratio of intervals, play an important role. In his paper "Tentamen novae theoriae musicae" (see [1]) of 1739, Euler defines the following arithmetic function ("Gradus-suavitalis function"). Let n be a positive integer and suppose its prime factorization is

$$n = p_1^{a_1} p_2^{a_2} \dots p_r^{a_r} \quad (p_i \text{ distinct primes, } a_i \geq 1)$$

Put

$$E(n) = 1 + \sum_{k=1}^r a_k (p_k - 1) \quad (1)$$

Let

$$E(1) = 1, \text{ by definitin}$$

In what follows, we will study this forgotten arithmetical function by Euler, but first note that for musical application Euler defined the function E also for the reduced fraction $\frac{x}{y}$ by

$$E\left(\frac{x}{y}\right) = E(x \cdot y)$$

Inserting fractions that represent ratios of musical intervals into his formula, we obtain the following values:

$$\begin{array}{ll} \text{octave :} & E\left(\frac{1}{2}\right) = 2 \\ \text{fifth :} & E\left(\frac{2}{3}\right) = 4 \\ \text{fourth :} & E\left(\frac{3}{4}\right) = 5 \\ \text{major third :} & E\left(\frac{4}{5}\right) = 7 \\ \text{minor third :} & E\left(\frac{5}{6}\right) = 8 \\ \text{major second :} & E\left(\frac{9}{10}\right) = 10 \\ \text{minor second :} & E\left(\frac{15}{16}\right) = 11 \\ \text{tritone :} & E\left(\frac{32}{45}\right) = 14 \end{array}$$

According to Euler, these numbers are a measure for the pleasantness of an interval; the smaller the value the more pleasing the interval. Indeed, this is more or less in accordance with our European listening habit, with one exception: the perfect fourth is heard as a dissonance in some contrapuntal and functional harmonic contexts (see [3]).

Remark. Euler used the notation $\Gamma(n)$ for his function, in place of $E(n)$. We have adopted this notation, as there is another important function introduced also by Euler in mathematics, the famous "Gamma function."

2. In what follows we will study the arithmetical function $E(n)$ of positive integers, defined by relation (1).

If the canonical factorization of n is $n = p_1^{a_1} \dots p_r^{a_r}$, then there are some well-known arithmetical functions, which are connected to the function $E(n)$.

Let $p(n), P(n)$ denote respectively the least and the greatest prime factors of n .

Let $\omega(n), \Omega(n)$ denote the number of distinct, respectively total number, of prime factors of n . Then clearly, $\omega(n) = r, \Omega(n) = a_1 + \dots + a_r$.

The following arithmetical function $B(n)$ has been intensively studied, too (see e.g. [4], [2]):

$$B(n) = \sum_{k=1}^r a_k p_k$$

Proposition 1. One has for $n > 1$

$$E(n) = 1 + B(n) - \Omega(n) \quad (2)$$

$$E(n) \geq 1 + \Omega(n) \quad (3)$$

Proof. Relation (2) is a consequence of (1) and the above introduced arithmetic functions. As

$$B(n) \geq \sum_{k=1}^r a_k \cdot 2 = 2\Omega(n),$$

inequality (3) follows by (2).

Proposition 2. For $n \geq 2$ one has the double inequality

$$1 + \Omega(n)(p(n) - 1) \leq E(n) \leq 1 + \Omega(n)(P(n) - 1) \quad (4)$$

Proof. Remark that

$$B(n) \leq \max\{p_1, \dots, p_r\} \sum_{k=1}^r a_k = P(n)\Omega(n),$$

and similarly

$$B(n) \geq \min\{p_1, \dots, p_r\} \sum_{k=1}^r a_k = P(n)\Omega(n)$$

From identity (2) we can deduce the double inequality (4).

Remark. (4) may be written also as

$$p(n) \leq 1 + \frac{E(n) - 1}{\Omega(n)} \leq P(n) \quad (5)$$

Remarking that $E(p) = 1 + (p - 1) = p$ for each prime p , one could ask for the fixed points of the function E .

Proposition 3. The fix points of the function E are only the prime numbers. In other words, one has

$$E(n) = n \text{ if } n = \text{prime}$$

Proof. We need the following two lemmas.

Lemma 1. $p^a \geq pa$ for all $p \geq 2$, $a \geq 1$; with equality only for $p = 2$, $a = 1$.

Proof. The inequality $p^{a-1} \geq a$ is true, as $p^{a-1} \geq 2^{a-1} \geq a$, which follows at once by mathematical induction.

Lemma 2. Let $x_i > 1$ ($i = \overline{1, r}$). Then one has

$$r + x_1 x_2 \dots x_r \geq 1 + x_1 + \dots + x_r \quad (6)$$

with equality only for $r = 1$.

Proof. For $r = 1$ there is equality; while for $r = 2$ the inequality is strict, as $2 + x_1 x_2 > 1 + x_1 + x_2$ by $(x_1 - 1)(x_2 - 1) > 0$, valid as $x_1 - 1 > 0$, $x_2 - 1 > 0$.

Assume now that (6) is true for $r \geq 2$ fixed, with a strict inequality. Then for $x_{r+1} > 1$ one has

$$\begin{aligned} r + 1 + x_1 x_2 \dots x_r x_{r+1} &> r + 1 + x_{r+1} (1 - r + x_1 + \dots + x_r) = \\ &= r - r x_{r+1} + (1 + x_{r+1} + x_1 + \dots + x_r) + (x_{r+1} - 1)(x_1 + \dots + x_r) > \\ &> 1 + x_1 + \dots + x_r + x_{r+1} \end{aligned}$$

as

$$r - r x_{r+1} + (x_{r+1} - 1)(x_1 + \dots + x_r) = (x_{r+1} - 1)(x_1 + \dots + x_r - r) > 0$$

as $x_1 + \dots + x_r > r$ and $x_{r+1} > 1$. By induction, we get that (6) is true for all r .

Proof of Proposition 3. One has

$$E(n) = 1 + a_1(p_1 - 1) + \dots + a_r(p_r - 1) \leq a_1 p_1 + \dots + a_r p_r - r + 1 \leq p_1^{a_1} \dots p_r^{a_r} = n,$$

with equality only for $r = 1$, $a_1 = 1$, i.e. when n is a prime. We have used Lemma 1 and Lemma 2.

Proposition 4. One has for $n \geq 2$,

$$E(n!) \leq 1 + n\pi(n), \quad (7)$$

where $\pi(n)$ denotes the number of all primes $\leq n$.

Proof. Let $n! = \prod_{p|n!} p^{a_p}$ be the prime factorization of $n!$.

By Legendre's theorem one has

$$a_p = \sum_{j=1}^{\infty} \left[\frac{n}{p^j} \right] \leq \sum_{j=1}^{\infty} \frac{n}{p^j} = \frac{n}{p} \left(1 + \frac{1}{p} + \frac{1}{p^2} + \dots \right) = \frac{n}{p} \cdot \frac{1}{1 - \frac{1}{p}} = \frac{n}{p-1}$$

Thus

$$E(n!) = 1 + \sum a_p (p-1)$$

where in the sum we have $\omega(n!)$ terms. Remark that $\omega(n!) = \pi(n)$, as in $n! = 1 \cdot 2 \cdot \dots \cdot n$ the number of distinct prime divisors is exactly the number of primes $\leq n$. As $a_p \leq \frac{n}{p-1}$, relation (7) follows.

Finally, we will obtain the overage order of the function $E(n)$:

Proposition 5. One has

$$\sum_{n \leq x} E(n) = \frac{\pi^2}{12} \cdot \frac{x^2}{\log x} + O\left(\frac{x^2}{\log^2 x}\right) \quad (8)$$

Proof. By the famous result of Hardy and Ramanujan (see e.g. [1]) one has

$$\sum_{n \leq x} \Omega(n) = x \log \log x + K \cdot x + O\left(\frac{x}{\log x}\right) \quad (9)$$

where K is a constant.

On the other hand, by a result of Alladi and Erdős (see [2], [4]) one has

$$\sum_{n \leq x} B(n) = \frac{\pi^2}{12} \cdot \frac{x^2}{\log x} + O\left(\frac{x^2}{\log^2 x}\right) \quad (10)$$

Now, by Proposition 1, relation (2) the expression (8) follows by remarking that

$$x \log \log x + K \cdot x = O\left(\frac{x^2}{\log^2 x}\right)$$

and

$$O\left(\frac{x}{\log x}\right) = O\left(\frac{x^2}{\log^2 x}\right)$$

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