NUMERICAL EVALUATION OF RELATIVISTIC SHOCK WAVES IN A MAGNETIZED PLASMA DOMINATED BY RADIATION

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ABSTRACT

For very hot fluids (in stars, in heavy nuclei, analysing fusion experiments) it is reasonable to apply the relativistic treatment and the strong discontinuities evolved in the medium have a great importance. Shocks of a magnetized relativistic plasma are examined supposing that the radiation term dominates the expressions of pressure and internal energy. From dynamical equations of the plasma we derive the jump conditions and introduce continuous scalars. In consequence of the simple state equation the key of the problem is to solve an irrational algebraic equation for the index of fluid. The shock amplitudes of density, pressure, magnetic induction and flow velocity will be determined. We produce the graphs showing the pressure and induction plotted against the shock speed for transversal and oblique external magnetic fields on various hydrodynamic and magnetic pressure numbers by numerical calculations. Similarly, we plot the transversal flow velocity arising behind the shock in an oblique magnetic field.

INTRODUCTION

In an earlier paper [1] Majorana and Anile studied magnetoacoustic shocks in a Synge gas. The relatively complicated caloric equation of state makes difficult the numerical evaluation. Conversely, in certain astrophysical models (e.g.: analysing supernova explosion) the medium can be considered as a radiation dominated plasma, and — thanks to the simple state equation — the volume of calculations decreases.

Our treatment stays within the framework of special relativity using Minkowski coordinates (x_4 = ict) and summation convention of repeated indices will be accepted. The thermodynamical quantities refer to the proper frame of medium. Let ρ , p, e, $\boldsymbol{v}$, respectively, the density, pressure, specific energy and flow velocity of the plasma. The world velocity is

$$\boldsymbol{\lambda} = \gamma\boldsymbol{v}/\mathrm{c} \quad , \quad \lambda_4 = \mathrm{i}\gamma$$

with $\gamma = (1-\boldsymbol{v}\cdot\boldsymbol{v}/c^2)^{-1/2}$; it seems that

$$\lambda_r \lambda_r = -1 \ .$$

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The electric intensity and magnetic induction will be denoted with $\boldsymbol{E}$ and $\boldsymbol{B}$. For the magnetic four-vector we have

$$\boldsymbol{b} = \gamma(\boldsymbol{B} + \boldsymbol{E} \times \boldsymbol{v}/c),\ \ b_4 = \mathrm{i}\gamma \boldsymbol{v}\cdot\boldsymbol{B}/c\ ;\ \lambda_r b_r = 0\ .$$

The plasma is regarded as a perfect fluid with an infinite electric conductivity, i.e. $\boldsymbol{E} = \boldsymbol{v} \times \boldsymbol{B}$, so

$$\boldsymbol{B} = \gamma(\boldsymbol{b} + \mathrm{i}b_4\boldsymbol{v}/c)$$

and

$$B^2 = \gamma^2 b^2 + b_4^2 \tag{1}$$

where $b^2 = b_r b_r$.

Let $\Phi\,(x_r)$ be a time-like regular hypersurface along which finite discontinuities in the quantities are permitted, and n_r be the unite normal four-vector to Φ. Then one may write [2]

$$\boldsymbol{n} = \Gamma\boldsymbol{N}\ ,\ n_4 = \mathrm{i}\Gamma V/c\ ;\ \Gamma = (1\text{-}V^2/c^2)^{-1/2}\ ,$$

where V is the shock speed and $\boldsymbol{N}$ represents the unite normal three-vector to the wave front printing to the region at rest (0) into which the shock enters from the perturbed region (1).

II. MAIN EQUATIONS

The main system of our problem contains the equation of continuity, the Maxwell equation of a perfect plasma, the equation of motion and the state equation. It is assumed that the particle number is conserved, so we have

$$\partial\left(\lambda\rho_i\right)/\,\partial x_i = 0\,, \tag{2}$$

on the other hand the Maxwell equation has the form

$$\partial\left(\lambda_i b_k - \lambda_k b_i\right)\partial x_k = 0. \tag{3}$$

The energy-moentum tensor can be written [3]

$$T_{ik} = \left(f\rho c^2 + b^2/\mu\right)\lambda_i\lambda_k + \left(p + b^2/2\mu\right)\delta_{ik} - b_i b_k/\mu \tag{4}$$

with

$$f = \left(e+p\right)/\rho c^2$$

(f is named index of fluid) for which it holds that

$$\partial T_{ik} / \partial x_k = 0. \tag{5}$$

The state equation of a medium dominated by radiation is [4]

$$e = \rho c^2 + 3p,$$

so the index assumes the form

$$f = 1 + 4p/\rho c^2 . \tag{6}$$

It is easy to see that such a plasma satisfies the Weil conditions, consequently a shock travelling in it is a compressive wave.

III. JUMP CONDITIONS. CONTINUOUS SCALARS

(2), (3) and (5) are conservation laws, hence we may write

$$[\rho\lambda_k] n_k = 0 , \; [\lambda_i b_k - \lambda_k b_i] n_k = 0 ,$$
$$\left[\left(f\rho c^2 + b^2/\mu\right)\lambda_i\lambda_k + \left(p + b^2/2\mu\right)\delta_{ik} - b_i b_k/\mu\right] n_k = 0$$

where $[\psi]$ denotes the jump of a quantity ψ across Φ :

$$[\psi] = \psi_1 - \psi_0$$

and μ is the vacuum permeability.

Let us introduce the abbreviations $\lambda_r\lambda_r = \theta$, $b_k n_k = \beta$. The quantities

$$J = \rho\theta ,$$
$$V_i = \beta\lambda_i - \theta\beta_i , \tag{7}$$
$$W_i = \left(f\rho c^2 + b^2/\mu\right)\theta\lambda_i + \left(p + b^2/2\mu\right)n_i - \beta b_i/\mu \tag{8}$$

are continuous along the shock front. From (7) and (8) one gets the following scalars:

$$H = V_i V_i = \theta^2 b^2 - \beta^2 ,$$
$$N = -V_i W_i / Jc^2 = f\beta ,$$
$$F = W_i n_i - H/\mu = c^2 f\rho\theta^2 + p + b^2/\mu ,$$
$$M = (F^2 - W_i W_i)/c^4 J^2 = f^2(\theta^2 + 1) + 2fb^2/\mu c^2\rho - 2pH/\mu c^4 J^2 .$$

The calculations will be carried out in the proper frame before the shock. By $V/c = u$ we have

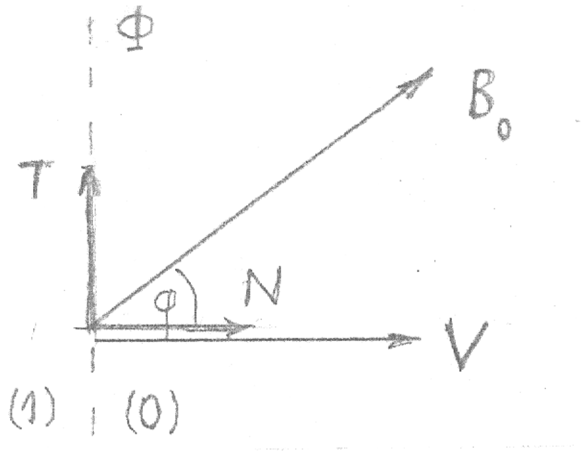

Fig.1
The shock front and the external magnetic field

$$\lambda_0 = \mathbf{0},\ \lambda_{40} = \mathrm{i},\ \theta_0 = \Gamma u\,,$$
$$b_{N0} = B_0 \cos\varphi\,,\ b_{T0} = B_0 \sin\varphi\,,\ b_{40} = 0\,,\quad \beta_0 = \Gamma B_0 \cos\varphi\,,$$

and

$$f_0 = 1 + 4s$$

where

$$s = p_0 / \rho_0 c^2$$

is the hydrodynamic pressure number, $\boldsymbol{T}$ denotes the unite tangential three-vector, moreover

$$\lambda_{N1} = \gamma_1 \xi\,,\ \lambda_{T1} = \gamma_1 \eta\,,\ \lambda_{41} = \mathrm{i}\gamma\,;$$
$$\xi = v_{N1} / c\,,\ \eta = v_{T1} / c\,.$$

IV. THE QUANTITIES BEHIND THE SHOCK

From the continuity of scalars J, N it is easy to get the density ratio and β_1:

$$r = \rho_1 / \rho_0 = -\Gamma u / \theta_1\,, \qquad \theta_1 = \gamma_1 \Gamma (\xi - u)\,, \tag{9}$$
$$\beta_1 = (f_0 / f_1)\, \Gamma B_0 \cos\varphi\,, \tag{10}$$

and the relation $[H]$=0 gives

$$b_1^2 / B_0^2 = (r^2 / u^2)\left[\left(f_0^2 / f_1^2\right)\cos^2\varphi + u^2 - \cos^2\varphi\right], \tag{11}$$

where we have used (9) and (10).

Now we consider that F has no jump. Inscribing (9) and (11) one finds

$$\left(\chi r^2/2u^2\right)\left[\left(f_0^2/f_1^2\right)\cos^2\varphi+u^2-\cos^2\varphi\right]+r\left(f_1-1\right)/4+u^2 f_1\Big/\left(1-u^2\right)r=$$

$$=u^2 f_0\Big/\left(1-u^2\right)+s+\chi/2\,, \tag{12}$$

where

$$\chi=B_0^{\,2}/\mu c^2\rho_0$$

is the magnetic pressure number.

From the condition $\left[M\right]=0$ we obtain

$$\left[u^2\Big/\left(1-u^2\right)r+1\right]f_1^2+\left(2\chi f_1 r/u^2\right)\left[\left(f_0^2/f_1^2\right)\cos^2\varphi+u^2-\cos^2\varphi\right]-$$
$$-\left(\chi r/2\right)\left(f_1-1\right)\left(u^2-\cos^2\varphi\right)\Big/u^2=f_0^2\Big/\left(1-u^2\right)+2\chi f_0-$$
$$-2s\chi\left(u^2-\cos^2\varphi\right)\Big/u^2\,. \tag{13}$$

Let us multiply (12) by f_1, (13) by r and subtract the arising expressions from one another: we get a quadratic equation for density ratio:

$$R=r/f_0\,,\qquad R=\left[-C\pm\left(C^2-2kD\right)^{1/2}\right]\Big/D$$

with

$$C=\left(3f_1^2+f_1\right)\Big/4-\left(1+4s\right)^2\Big/\left(1-u^2\right)-2\chi\left[1+\left(3+u^{-2}\cos^2\varphi\right)s\right],$$
$$D=\chi\left[\left(2f_1^2+f_1\right)\left(1-u^{-2}\cos^2\varphi\right)+3\left(1+4s\right)^2u^{-2}\cos^2\varphi\right],$$
$$k=\chi/2+\left[\left(1+3s\right)u^2+s\right]\Big/\left(1-u^2\right). \tag{14}$$

Putting (14) into (12) we have an irrational algebraic equation for the index of fluid behind the shock which contains the parameters u,φ,s and χ:

$$2\chi R^3\left[\left(1-u^{-2}\cos^2\varphi\right)f_1^2+\left(1+4s\right)^2u^{-2}\cos^2\varphi\right]+R^2\left(f_1^2-f_1\right)-$$
$$-4kR+4u^2\Big/\left(1-u^2\right)=0\,. \tag{15}$$

With knowledge of f_1 (14) gives r, while the pressure ratio $q=p_1/p_0$ can be read off from (6):

$$q = (f_1 - 1)r/4s\,.$$

Now we are going to compute the jump of flow velocity. Owing to continuity of V_T

$$b_{T1}/B_0 = r\left[\sin\varphi - (f_0/f_1 u)\gamma_1\eta\cos\varphi\right], \tag{16}$$

and from the condition $[W_T] = 0$ one gets

$$\left(rf_1/\chi + b_1^2/b_0^2\right)(\Gamma u/r)\gamma_1\eta + \beta_1 b_{T1}/B_0^2 = \mu\Gamma\sin\varphi\cos\varphi\,. \tag{17}$$

Substituting (16) into (17) and making use of (10), (11) one finds

$$g = \gamma_1\eta = \left\{2uf_1\left[\chi^{-1} + R\left(1 - u^{-2}\cos^2\varphi\right)\right]\right\}^{-1}\left[1 - (1+4s)R\right]\sin 2\varphi\,. \tag{18}$$

Let us employ the identity $\gamma_1^2\left(1 - \xi^2 - \eta^2\right) = 1$ and the relation

$$\gamma_1(u - \xi) = u/r$$

got from (9). Taking account of (18) we obtain the expression of Lorentz factor behind the shock:

$$\gamma_1 = \left\{\left[\left(u^2/r\right)^2 + \left(1-u^2\right)\left(1 + u^2/r + g^2\right)\right]^{1/2} - u^2/r\right\}\Big/\left(1-u^2\right). \tag{19}$$

Now then the jump of normal flow velocity is

$$\xi = u\left(1 - 1/r\gamma_1\right),$$

moreover η is given by (18), (19).

It still remains to be done the expression of induction ratio $h = B_1/B_0$. From the condition $[V_4] = 0$, with an eye to (10), we have

$$b_{41}/B_0 = \mathrm{i}(r/u)\left(1 - \gamma_1 f_0/f_1\right)\cos\varphi\,, \tag{20}$$

while (1) leads to the expression

$$h^2 = \gamma_1^2\left(b_1/B_0\right)^2 + \left(b_{41}/B_0\right)^2.$$

After substituting (11) and (20) we find

$$h = r\left\{\gamma_1^2\left(1-u^{-2}\cos^2\varphi\right)+\left[2\left(1+4s\right)\gamma_1/f_1-1\right]u^{-2}\cos^2\varphi\right\}^{1/2}.$$

Lichnerowicz has shown [3] that in a compressive shock $[f]$, $[\rho]$ and $[p]$ are positive quantities, so

$$f_1 > 1+4s\,,\ R > 1/f_1\,. \tag{21}$$

V. NUMERICAL EVALUATION

The equation (15) has been solved for the values $\pi/2$ and $\pi/4$ of φ by numerical computations. For the parameter pair (s,χ) the values (0.1, 0.1), (0.1, 0.2) and (0.5, 0.1) have been taken.

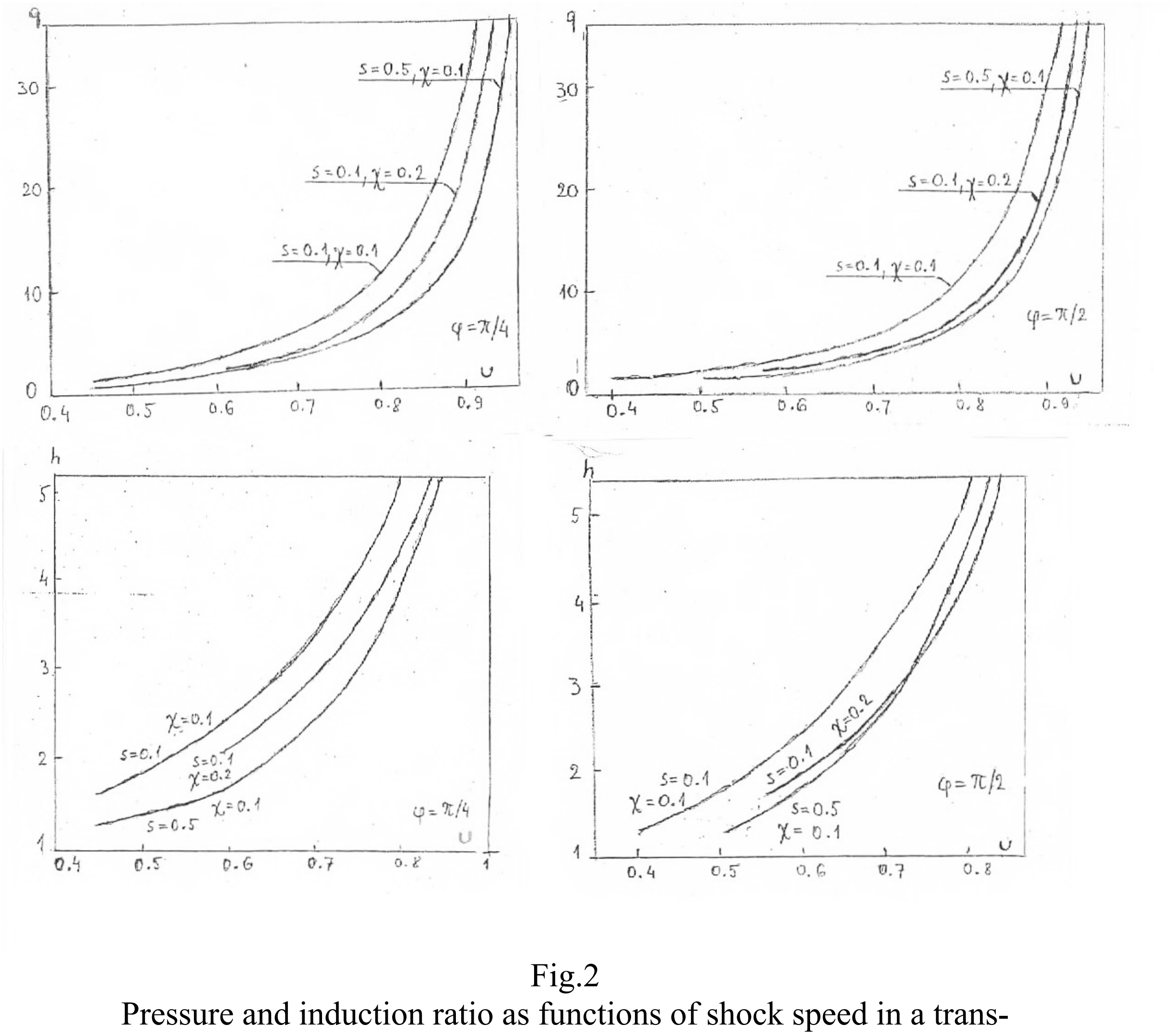

Fig.2
Pressure and induction ratio as functions of shock speed in a transversal and oblique magnetic field for various parameter pairs (s,χ)

From the solutions we have chosen the pairs (f, R) satisfying the criteria (21). With knowledge of these it is easy to compute the values of q, b and η (if the external magnetic field is longitudinal or transversal, η vanishes). These quantities have been plotted against the shock speed.

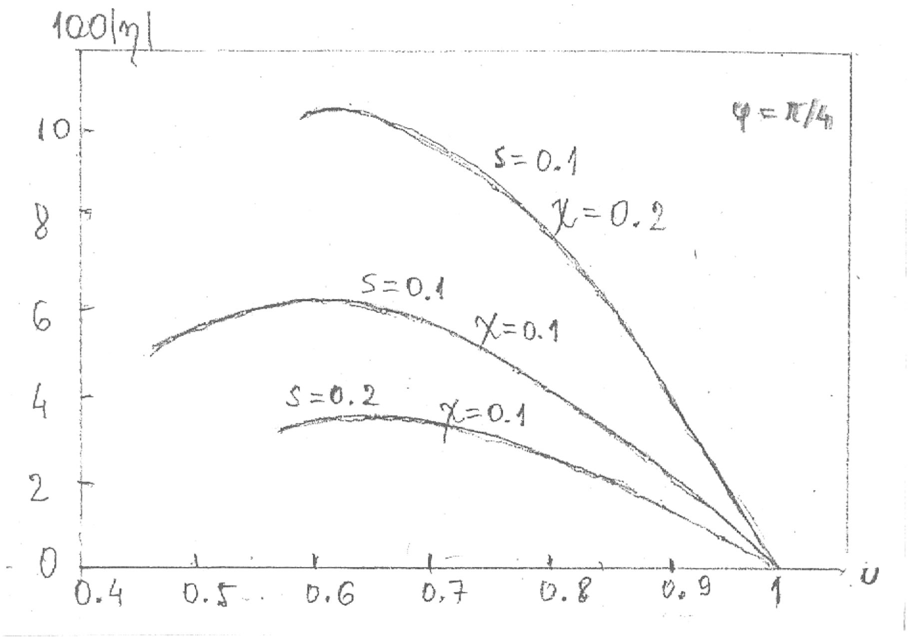

Fig.3
Transversal flow velocity behind the shock as function of shock speed in an oblique magnetic field $(\varphi = \pi/4)$ for various parameter pairs (s, χ)

In the case of a Synge gas these functions have two branches if the magnetic field is oblique [4]. In our case the lower branch does not exist: for h^2 one obtain negative values.

Intensifying the external magnetic field displaces the minimal shock velocity required to form a shock in the direction of light speed in vacuum. The transversal flow velocity is not a monotonic function of *u*, and it vanishes if *V* approaches *c*.

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