A NONLINEAR KINEMATICAL MODEL FOR HETEROGENEOUS CIRCULAR BEAMS

***György Szeidl*[1], *László Kiss*[2]**
[1]Professor Emeritus, [2] Assistant Lecturer
Institute of Applied Mechanics, University of Miskolc
3515 Miskolc-Egyetemvros, Hungary

1. INTRODUCTION

In this article a new kinematical model for heterogeneous circular beams is established. The derivation process follows the steps gathered in [1] by Kozák for shells. This model uses less neglects compared to the reviewed literature and provide base for a future stability model using a finite element (FE) algorithm. Regarding the preliminaries, article [2] by Dawe investigates the stability of deep and shallow circular beams using FE code. The author considers the linearized theory. Flores and Godoy in [3] discretize 3D continuums to determine the critical load both for limit point and bifurcation buckling. Ascione and Fraternali [4] assume geometrically nonlinear behaviour (the rotation and angle distortion are relatively great and the strain is infinitesimally small) and arbitrary curvature for bimodular curved beams. Pi and Trahair [5] also present a nonlinear FE model. They account for the pre-buckling strains and keep second-order terms in the curvature change and bending strain. It is a common property of the papers cited that the terms involving (a) the product of the axial strain and the rotation as well as (b) the square of the rotation field are all neglected when setting up a kinematical model.

2. THE INITIAL CONFIGURATION $\mathfrak{B}$

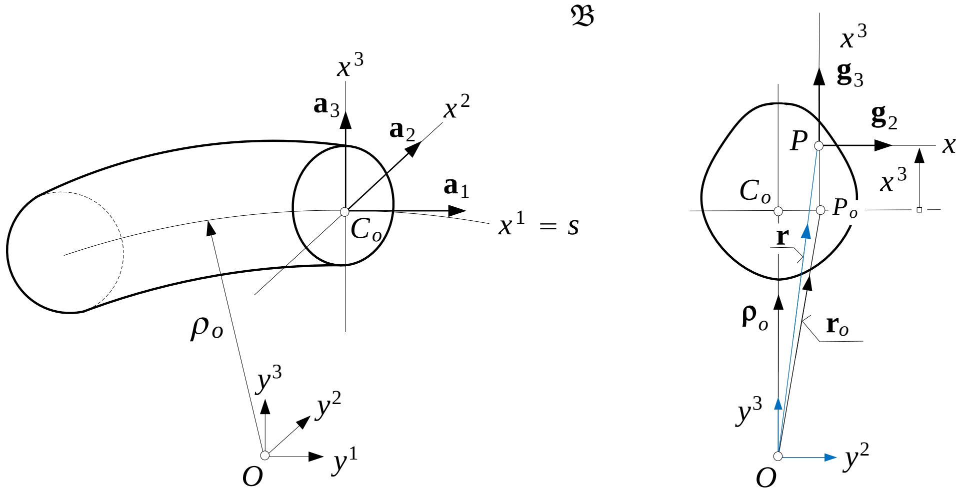

Figure 1. Circular beam in the initial configuration

Figure 1 presents a circular beam with uniform cross-section in the initial configuration ($\mathfrak{B}$). The origin of the identifier coordinate system is denoted by O, the coordinate axes (using index notation) are y^i and the base is formed by the unit vectors $\mathbf{e}_i$. Latin indexes can take on values of $1, 2$ and 3 (except for $_o$ which denotes that quantities are measured on the surface $x^3 = 0$), while Greek indexes can be 1 and 2

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only. The plane y^1y^3 is a symmetry plane. Young's modulus is identified with a letter E. Its distribution is symmetric to the axis x^3, that is, $E(x^2,x^3)=E(-x^2,x^3)$. C_o is the E-weighted centroid of the illustrated cross-section, where the E-weighted first moment with respect to the axis x^2 vanishes

$$\mathcal{Q}_{o2}=\int_A x^3 E\mathrm{dA}=0\,, \tag{1}$$

and the axis intersecting these points is referred to as the (E-weighted) centerline, whose radius is constant and denoted by ρ_o. Along this line, we measure the arc coordinate $x^1=s$. The coordinate surface $x^2=0$ is the symmetry plane of the beam. The coordinate system x^i moves together with the beam. A subscript preceded by the symbols [,](;) mean [partial derivative](covariant derivative) with respect to the corresponding coordinate. Further, δ_k^ℓ is the Kronecker delta, e_{pqr} is the covariant- and $e^{k\ell m}$ is the contravariant permutation symbol. The permutation tensors satisfy the relations

$$\epsilon_{pqr}=\sqrt{g}e_{pqr}\,,\qquad \epsilon^{k\ell m}=e^{k\ell m}/\sqrt{g}, \tag{2}$$

where $g=|g_{k\ell}|$ is the determinant of the metric tensor $g_{k\ell}$. The covariant base vectors on the surface $x^3=0$ are denoted by $\mathbf{a}_i$, while at any other point $\mathbf{g}_i$ give the base. Recalling Figure 1, at P_o, we have

$$\mathbf{a}_1=\frac{\partial\mathbf{r}_o}{\partial x^1}=\mathbf{r}_{o,1}=\boldsymbol{\rho}_{o,1};\;\; \mathbf{a}_2=\mathbf{r}_{o,2}=\mathbf{e}_2\,,\;\mathbf{a}_3=\frac{\mathbf{a}_1\times\mathbf{a}_2}{|\mathbf{a}_1\times\mathbf{a}_2|}=-\frac{1}{\rho_o\left(s\right)}\mathbf{a}_{1,1}. \tag{3}$$

As $x^1=s$ these are all unit vectors and coincide with the contarvariant base vectors $\mathbf{a}^i$. Thus the metric tensor a_{kl} on the surface $x^3=0$ is a unit tensor and its determinant a is 1. In any arbitrary point P the we find that

$$\mathbf{g}_1=\mathbf{r}_{,1}=\mathbf{a}_1+\mathbf{a}_{3,1}x^3\,,\quad \mathbf{g}_2=\mathbf{r}_{,2}=\mathbf{a}_2\,,\quad \mathbf{g}_3=\mathbf{r}_{,3}=\mathbf{a}_3. \tag{4}$$

Therefore, we can now introduce the b_α^β curvature tensor as

$$\mathbf{g}_\alpha=\mathbf{r}_{,\alpha}=\mu_\alpha^{\ \beta}\mathbf{a}_\beta=\mathbf{a}_\alpha-b_\alpha^\beta\mathbf{a}_\beta x^3\,,\;\text{ with }\;\mathbf{a}_{3,\alpha}=-b_\alpha^\beta\mathbf{a}_\beta\;\text{ and }\;\left[b_\alpha^\beta\right]=\begin{bmatrix}-\frac{1}{\rho_o} & 0\\ 0 & 0\end{bmatrix}. \tag{5}$$

The inverse transformation tensor is approximated by its power series

$$\left[\overset{-1}{\mu}{}_\beta^{\ \kappa}\right]=\begin{bmatrix}1&0\\0&1\end{bmatrix}+\begin{bmatrix}-\frac{1}{\rho_o}&0\\0&1\end{bmatrix}x^3+\begin{bmatrix}\left(\frac{1}{\rho_o}\right)^2&0\\0&1\end{bmatrix}\left(x^3\right)^2+\begin{bmatrix}-\left(\frac{1}{\rho_o}\right)^3&0\\0&1\end{bmatrix}\left(x^3\right)^3+\cdots\,. \tag{6}$$

With (4) one can construct the product $g_{k\ell}=\mathbf{g}_k\cdot\mathbf{g}_\ell$:

$$[g_{\alpha\beta}]=\begin{bmatrix}\left(1+\frac{x^3}{\rho_o}\right)^2&0\\0&1\end{bmatrix}=\begin{bmatrix}1&0\\0&1\end{bmatrix}+\begin{bmatrix}\frac{2}{\rho_o}&0\\0&1\end{bmatrix}x^3+\begin{bmatrix}\frac{1}{\rho_o^2}&0\\0&1\end{bmatrix}\left(x^3\right)^2. \tag{7}$$

The determinant of the former metric tensor is

$$|g_{kl}|=g=\left(1+\frac{x^3}{\rho_o}\right)^2. \tag{8}$$

3. THE PRESENT CONFIGURATION $\bar{\mathfrak{B}}$

The typical quantities in this configuration are denoted by a bar symbol, as it can be observed in Figure 2. The $\mathbf{u}_o$ displacement vector of C_o is given by

$$\mathbf{u}_o\left(x^1\right) = u_o^1\left(x^1\right)\mathbf{a}_1 + u_o^3\left(x^1\right)\mathbf{a}_3 \tag{9}$$

and so the base vectors of the deformed mid-surface are

$$\bar{\mathbf{a}}_1 = \bar{\mathbf{r}}_{o,1} = \mathbf{a}_1 + \mathbf{u}_{o,1} = \left(\delta_1^l + u_{o;1}^l\right)\mathbf{a}_l,\ \bar{\mathbf{a}}_2 = \bar{\mathbf{r}}_{o,1} = \mathbf{a}_2,\ \bar{\mathbf{a}}_3 = \frac{\bar{\mathbf{a}}_1 \times \bar{\mathbf{a}}_2}{|\bar{\mathbf{a}}_1 \times \bar{\mathbf{a}}_2|} = \bar{\mathbf{n}}. \tag{10}$$

We remark that here

$$u_{o;k}^l = u_{o,k}^l + u_o^s\Gamma_{sk}^l\,, \quad |\bar{\mathbf{a}}_1 \times \bar{\mathbf{a}}_2| = (\bar{\mathbf{a}}_1 \times \bar{\mathbf{a}}_2)\cdot\bar{\mathbf{a}}_3 = \bar{\epsilon}_{o\,123} = \sqrt{\bar{a}}\,, \tag{11}$$

with Γ_{sk}^l being the Christoffel symbols of the second kind in the applied cylindrical coordinate system. Recalling these from the literature we get

$$\bar{\mathbf{a}}_1 = \left(1 + u_{o,1}^1 + \frac{u_o^3}{\rho_o}\right)\mathbf{a}_1 + \left(u_{o,1}^3 - \frac{u_o^1}{\rho_o}\right)\mathbf{a}_3\,. \tag{12}$$

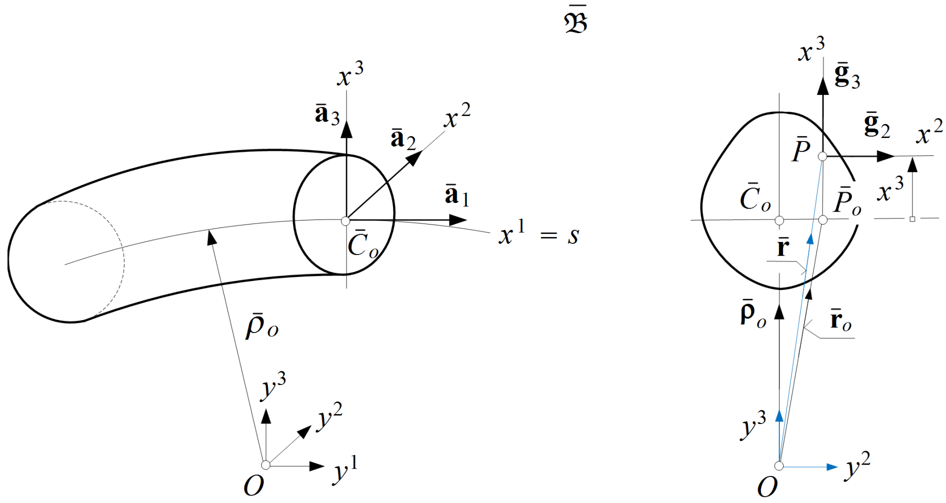

Figure 2. Circular beam in the present configuration

The components of the metric tensor $\bar{a}_{k\ell}$ are $\bar{a}_{22} = \bar{a}_{33} = 1\,, \bar{a}_{\beta 3} = a_{3\beta} = 0$ and

$$\bar{a}_{11} = \bar{\mathbf{a}}_1\cdot\bar{\mathbf{a}}_1 = 1 + 2\left(u_{o,1}^1 + \frac{u_o^3}{\rho_o}\right) + \left(u_{o,1}^1 + \frac{u_o^3}{\rho_o}\right)^2 + \left(u_{o,1}^3 - \frac{u_o^1}{\rho_o}\right)^2. \tag{13a}$$

This latter coordinate at the same time coincides with the determinant $\bar{a}$.

For our latter considerations we now continue with the projections of the base vectors in the present configuration onto those in the initial configuration. To do so, we recall (3), (10) and introduce the notational convention $\bar{\mathbf{a}}_i\cdot\mathbf{a}_j = d_{ij}$ the matrix of which is of the form

$$[d_{ij}] = \begin{bmatrix} 1 + u_{o,1}^1 + \frac{u_o^3}{\rho_o} & 0 & u_{o,1}^3 - \frac{u_o^1}{\rho_o} \\ 0 & 1 & 0 \\ -\frac{1}{\sqrt{\bar{a}}}\left(u_{o,1}^3 - \frac{u_o^1}{\rho_o}\right) & 0 & \frac{1}{\sqrt{\bar{a}}}\left(1 + u_{o,1}^1 + \frac{u_o^3}{\rho_o}\right) \end{bmatrix}. \tag{14}$$

4. THE GREEN-LAGRANGE STRAIN TENSOR

According to the definition this tensor has the form

$$\hat{E}_{ab} = \frac{1}{2}\left(\bar{g}_{ab} - g_{ab}\right) \tag{15}$$

at point P and can be transformed onto the surface $x^3 = 0$ using (6) which yields

$$E_{k\ell} = \overset{-1}{\mu}{}^{a}_{k}\,\overset{-1}{\mu}{}^{b}_{\ell}\,\hat{E}_{ab}. \tag{16}$$

We therefore need the coordinates $\bar{g}_{ab} = \bar{\mathbf{g}}_a \cdot \bar{\mathbf{g}}_b$. At this point we must make a kinematical assumption. We assume the validity of the Kirchhoff hypothesis. Because of this the position vector after deformation in the present configuration is

$$\bar{\mathbf{r}} = \boldsymbol{\rho}_o + \mathbf{u}_o + x^2\bar{\mathbf{a}}_2 + x^3\bar{\mathbf{a}}_3 = \bar{\mathbf{r}}_o + x^3\bar{\mathbf{a}}_3\ . \tag{17}$$

The covariant base vectors $\bar{\mathbf{g}}_\alpha$ in $\bar{P}$ are

$$\bar{\mathbf{g}}_1 = \bar{\mathbf{r}}_{,1} = \bar{\mathbf{a}}_1 + x^3\bar{\mathbf{a}}_{3,1} = \mathbf{a}_1 + \mathbf{u}_{o,1} + x^3\bar{\mathbf{a}}_{3,1}\ ,\ \bar{\mathbf{g}}_2 = \mathbf{a}_2\ ,\ \bar{\mathbf{g}}_3 = \bar{\mathbf{a}}_3\ . \tag{18}$$

The restrictions due to the hypothesis are: (a) $\mathbf{u}_o = \mathbf{u}_o(x^1)$ and (b) according to (10), $\bar{\mathbf{a}}_3 = \bar{\mathbf{a}}_3(x^1)$.

If we compare (15) and (18) it can be shown that the Green-Lagrange strain tensor $\boldsymbol{E}$ consists of terms up until the second power of the x_3 coordinate:

$$\hat{E}_{\alpha\beta} = \mathbf{g}_\alpha \cdot \boldsymbol{E} \cdot \mathbf{g}_\beta = \overset{0}{\hat{E}}_{\alpha\beta} + \overset{1}{\hat{E}}_{\alpha\beta}x^3 + \overset{2}{\hat{E}}_{\alpha\beta}\left(x^3\right)^2 \tag{19}$$

with

$$\overset{0}{\hat{E}}_{\alpha\beta} = \frac{1}{2}\left(\bar{a}_{\alpha\beta} - \delta_{\alpha\beta}\right),\ \overset{1}{\hat{E}}_{\alpha\beta} = \frac{1}{2}\left(\bar{\mathbf{a}}_{3,\alpha}\cdot\bar{\mathbf{a}}_\beta + \bar{\mathbf{a}}_\alpha\cdot\bar{\mathbf{a}}_{3,\beta} - \overset{1}{g}_{\alpha\beta}\right),\ \overset{2}{\hat{E}}_{\alpha\beta} = \frac{1}{2}\left(\bar{\mathbf{a}}_{3,\alpha}\cdot\bar{\mathbf{a}}_{3,\beta} - \overset{2}{g}_{\alpha\beta}\right). \tag{20}$$

For the considered beam problem, the component $\hat{E}_{11}$ is of particular importance. Plugging (10) into (20) yields the terms in question. The first one is

$$\overset{0}{\hat{E}}_{11} = \frac{1}{2}\left[\left(\mathbf{a}_1 + u^l_{o;1}\mathbf{a}_l\right)\cdot\left(\mathbf{a}_1 + u^k_{o;1}\mathbf{a}_k\right) - a_{11}\right] = \underbrace{\left(u^1_{o,1} + \frac{u^3_o}{\rho_o}\right)}_{\varepsilon_{o1}} + \frac{1}{2}\underbrace{\left(u^1_{o,1} + \frac{u^3_o}{\rho_o}\right)}_{\varepsilon_{o1}}^2 +$$

$$+\frac{1}{2}\underbrace{\left(u^3_{o,1} - \frac{u^1_o}{\rho_o}\right)}_{-\psi_{o2}}^2 = \varepsilon_{o1} + \frac{1}{2}\left(\varepsilon_{o1}\right)^2 + \frac{1}{2}\left(-\psi_{o2}\right)^2 \simeq \varepsilon_{o1} + \frac{1}{2}\left(\psi_{o2}\right)^2\ . \tag{21}$$

In the latter formula ε_{o1} is the membrane strain on the surface $x^3 = 0$ (centerline) according to the classical linear theory and ψ_{o2} is the linearized rotation there. Whilst scientific articles like [6, 7], in general, solely keep those second-order terms which are in relation with the rotation, we intend to not keep only the square of the membrane strain ε_{o1}. The term, linear in x^3 follows after a substitution to $(20)_2$:

$$\overset{1}{\hat{E}}_{11} = \frac{1}{2}\left(\mathbf{a}_1 + u^l_{o;1}\mathbf{a}_l\right)\cdot\bar{\mathbf{a}}_{3,1} + \frac{1}{2}\left(\mathbf{a}_1 + u^k_{o;1}\mathbf{a}_k\right)\cdot\bar{\mathbf{a}}_{3,1} - \frac{1}{\rho_o}. \tag{22}$$

We shall clarify the calculation of $\bar{\mathbf{a}}_{3,1}$ using the form $\bar{\mathbf{a}}_3 = d_{o\,m}\mathbf{a}^m$ for $\bar{\mathbf{a}}_3$. The coordinates $d_{o\,m}$ can be obtained from (14) as

$$\bar{\mathbf{a}}_{3,\alpha} = d_{o\,m;\alpha}\mathbf{a}^m = \left(d_{o\,m,\alpha} - \Gamma^s_{\alpha m}d_{o\,s}\right)\mathbf{a}^m\ . \tag{23}$$

Therefore we have

$$\overset{1}{\hat{E}}_{11} = \left(1 + u^1_{o,1} + \frac{u^3_o}{\rho_o}\right)\left(d_{o\,1,1} + \frac{d_{o\,3}}{\rho_o}\right) + \left(u^3_{o,1} - \frac{u^1_o}{\rho_o}\right)\left(d_{o\,3,1} - \frac{d_{o\,1}}{\rho_o}\right) - \frac{1}{\rho_o}\,. \quad (24)$$

We shall now mention that in the expressions of the components $d_{o\,1}, d_{o\,3}$ – see (14) – we approximated $1/\sqrt{a}$ using only the first two terms of (13a), yielding

$$\frac{1}{\sqrt{a}} \simeq 1 - \varepsilon_{o1} - \frac{1}{2}\left(\psi_{o2}\right)^2. \quad (25)$$

Consequently

$$\overset{1}{\hat{E}}_{11} \simeq \frac{\varepsilon_{o1}}{\rho_o} + \psi_{o2,1} + \frac{1}{2\rho_o}\left(\psi_{o2}\right)^2 - \psi_{o2}\varepsilon_{o1,1}. \quad (26)$$

Finally, from $(20)_3$ we get

$$\overset{2}{\hat{E}}_{11} = \frac{1}{\rho_o}\psi_{o2,1} + \frac{1}{2}\left(\psi_{o2,1}\right)^2 - \frac{\varepsilon_{o1}}{\rho_o}\psi_{o2,1} - \frac{\psi_{o2}}{\rho_o}\varepsilon_{o1,1}. \quad (27)$$

To summarize the former outcomes, equations (21), (24) and (27) together give the membrane strain from (19) in the base of P:

$$\hat{E}_{11} = \varepsilon_{o1} + \frac{1}{2}\left(\psi_{o2}\right)^2 + x^3\left[\frac{\varepsilon_{o1}}{\rho_o} + \psi_{o2,1} + \frac{1}{2\rho_o}\left(\psi_{o2}\right)^2 - \psi_{o2}\varepsilon_{o1,1}\right] + \\ + (x^3)^2\left[\frac{1}{\rho_o}\psi_{o2,1} + \frac{1}{2}\left(\psi_{o2,1}\right)^2 - \frac{\varepsilon_{o1}}{\rho_o}\psi_{o2,1} - \frac{\psi_{o2}}{\rho_o}\varepsilon_{o1,1}\right]. \quad (28)$$

Let us introduce now the notation

$$\frac{\mathrm{d}}{\mathrm{d}x^1}(\cdots) = \frac{\mathrm{d}}{\mathrm{d}s}(\cdots) = (\cdots)_{,1} = \frac{1}{\rho_o}\frac{\mathrm{d}}{\mathrm{d}\varphi} = \frac{1}{\rho_o}(\cdots)^{(1)} \quad (29)$$

for the derivatives where φ is the angle coordinate ($s = \rho_o\varphi$). Thus

$$\hat{E}_{11} = \underbrace{\varepsilon_{o1} + \frac{1}{2}\left(\psi_{o2}\right)^2}_{\overset{0}{\hat{e}}_{11}} + \frac{x^3}{\rho_o}\underbrace{\left[\varepsilon_{o1} + \psi^{(1)}_{o2} + \frac{1}{2}\left(\psi_{o2}\right)^2 - \psi_{o2}\varepsilon^{(1)}_{o1}\right]}_{\overset{1}{\hat{e}}_{11}} + \\ + \left(\frac{x^3}{\rho_o}\right)^2\underbrace{\left[\psi^{(1)}_{o2} + \frac{1}{2}\left(\psi^{(1)}_{o2}\right)^2 - \varepsilon_{o1}\psi^{(1)}_{o2} - \psi_{o2}\varepsilon^{(1)}_{o1}\right]}_{\overset{2}{\hat{e}}_{11}} = \overset{0}{\hat{e}}_{11} + \frac{x^3}{\rho_o}\overset{1}{\hat{e}}_{11} + \left(\frac{x^3}{\rho_o}\right)^2\overset{2}{\hat{e}}_{11}\,. \quad (30)$$

One final practical task is to transform this strain onto the surface $x^3 = 0$. We can express it in the base $\mathbf{a}_\ell$ using

$$E_{11} = \overset{-1}{\mu}{}^{\kappa}_{1}\,\hat{E}_{\kappa\lambda}\,\overset{-1}{\mu}{}^{\lambda}_{1}. \quad (31)$$

Accordingly

$$E_{11} = \overset{0}{e}_{11} + \frac{x^3}{\rho_o}\overset{1}{e}_{11} + \left(\frac{x^3}{\rho_o}\right)^2\overset{2}{e}_{11} \quad (32)$$

is the desired form of the strain component, where

$$\overset{0}{e}_{11} = \overset{0}{\hat{e}}_{11}\,,\ \ \overset{1}{e}_{11} = -2\overset{0}{\hat{e}}_{11} + \overset{1}{\hat{e}}_{11}\,,\ \ \overset{2}{e}_{11} = 3\overset{0}{\hat{e}}_{11} - 2\overset{1}{\hat{e}}_{11} + \overset{2}{\hat{e}}_{11}\,. \quad (33a)$$

Therefore

$$\overset{0}{e}_{11} = \varepsilon_{o1} + \frac{1}{2}(\varepsilon_{o1})^2 + \frac{1}{2}(-\psi_{o2})^2 \simeq \varepsilon_{o1} + \frac{1}{2}(\psi_{o2})^2 \ , \tag{34a}$$

$$\overset{1}{e}_{11} \simeq \psi_{o2}^{(1)} - \varepsilon_{o1} - \frac{1}{2}(\psi_{o2})^2 - \psi_{o2}\varepsilon_{o1}^{(1)} \ , \tag{34b}$$

$$\overset{2}{e}_{11} = \varepsilon_{o1} - \psi_{o2}^{(1)} + \frac{1}{2}\left(\psi_{o2}^{(1)}\right)^2 - \varepsilon_{o1}\psi_{o2}^{(1)} + \psi_{o2}\varepsilon_{o1}^{(1)} + \frac{1}{2}(\psi_{o2})^2 \ . \tag{34c}$$

Altogether – again, neglecting the terms above $(x_3)^2$ – we have

$$E_{11} = \left[\varepsilon_{o1} + \frac{1}{2}(\psi_{o2})^2\right] + \frac{x^3}{\rho_o}\left[\psi_{o2}^{(1)} - \varepsilon_{o1} - \frac{1}{2}(\psi_{o2})^2 - \psi_{o2}\varepsilon_{o1}^{(1)}\right] + \\ + \left(\frac{x^3}{\rho_o}\right)^2\left[\varepsilon_{o1} - \psi_{o2}^{(1)} + \frac{1}{2}\left(\psi_{o2}^{(1)}\right)^2 - \varepsilon_{o1}\psi_{o2}^{(1)} + \psi_{o2}\varepsilon_{o1}^{(1)} + \frac{1}{2}(\psi_{o2})^2\right] . \tag{35}$$

We should decompose this result to a linear and a nonlinear part:

$$E_{11} = E_{11}^L + E_{11}^N \ , \qquad E_{11}^L = \frac{1}{1+\frac{x^3}{\rho_o}}\left(\varepsilon_{o1} + \frac{x^3}{\rho_o}\psi_{o2}^{(1)}\right) \tag{36a}$$

and

$$E_{11}^N = \frac{1}{1+\frac{x^3}{\rho_o}}\frac{1}{2}(\psi_{o2})^2 - \frac{x^3}{\rho_o}\psi_{o2}\varepsilon_{o1}^{(1)} + \left(\frac{x^3}{\rho_o}\right)^2\left[\frac{1}{2}\left(\psi_{o2}^{(1)}\right)^2 - \varepsilon_{o1}\psi_{o2}^{(1)} + \psi_{o2}\varepsilon_{o1}^{(1)}\right] \ . \tag{36b}$$

This achievement is comparable with the models by the likes of Bateni [6], Bradford et al. [7], Kiss & Szeidl [8] which, in general, apply either

$$E_{11} = \varepsilon_{o1} + \frac{x^3}{\rho_o}\psi_{o2}^{(1)} + \frac{1}{2}(\psi_{o2})^2 \ , \tag{37a}$$

or

$$E_{11} \simeq \left[\varepsilon_{o1} + \frac{1}{2}(\psi_{o2})^2\right] + \frac{x^3}{\rho_o}\left[\psi_{o2}^{(1)} - \varepsilon_{o1}\right] + \left(\frac{x^3}{\rho_o}\right)^2\left[\varepsilon_{o1} - \psi_{o2}^{(1)}\right] \tag{37b}$$

for the approximation of the axial strain of the centerline. The differences between the new and former models are now easily noticeable.

5. THE INNER FORCES

Here we present some possible formulation expressing the inner forces. The constitutive equation is $S_{11} = E(x^2, x^3)\,E_{11}$ where S_{11} is a component of the second Piola-Kirchhoff stress tensor. Further, under the assumption of cross-sectional inhomogeneity, Young's modulus satisfies the relation $E(x^2, x^3) = E(-x^2, x^3)$. In the proceeding, we provide some approximations.

We commence with the simplest linearized model. In this case the strain and stress in the axial direction are

$$E_{11}^L = \frac{1}{1+\frac{x^3}{\rho_o}}\left(\varepsilon_{o1} + \frac{x^3}{\rho_o}\psi_{o2}^{(1)}\right) \ , \qquad S_{11} = \frac{1}{1+\frac{x^3}{\rho_o}}\left(\varepsilon_{o1} + \frac{x^3}{\rho_o}\psi_{o2}^{(1)}\right)E\left(x^2, x^3\right) \ , \tag{38}$$

which means that the axial force N and the bending moment M have the forms

$$N=\int_A S_{11}\,\mathrm{d}A=\int_A \frac{E\left(x^2,x^3\right)}{1+\frac{x^3}{\rho_o}}\,\mathrm{d}A\,\varepsilon_{o1}+ \\ +\frac{1}{\rho_o}\int_A \frac{x^3 E\left(x^2,x^3\right)}{1+\frac{x^3}{\rho_o}}\,\mathrm{d}A\,\psi_{o2}^{(1)}=A_{e\rho}\varepsilon_{o1}+\frac{Q_{e\rho}}{\rho_o}\psi_{o2}^{(1)} \quad (39a)$$

and

$$M=\int_A x^3 S_{11}\,\mathrm{d}A=\int_A \frac{x^3 E\left(x^2,x^3\right)}{1+\frac{x^3}{\rho_o}}\,\mathrm{d}A\,\varepsilon_{o1}+ \\ +\frac{1}{\rho_o}\int_A \frac{\left(x^3\right)^2 E\left(x^2,x^3\right)}{1+\frac{x^3}{\rho_o}}\,\mathrm{d}A\,\psi_{o2}^{(1)}=Q_{e\rho}\varepsilon_{o1}+\frac{I_{e\rho}}{\rho_o}\psi_{o2}^{(1)}\,. \quad (39b)$$

The definition of the newly introduced quantities like $A_{e\rho}; Q_{e\rho}$ and $I_{e\rho}$ can be understood from the provided formulae.

A bit more accurate approximation keeps the the square of the rotation field. Thus

$$E_{11}=\frac{1}{1+\frac{x^3}{\rho_o}}\left(\varepsilon_{o1}+\frac{x^3}{\rho_o}\psi_{o2}^{(1)}+\frac{1}{2}\left(\psi_{o2}\right)^2\right),\quad S_{11}=\frac{1}{1+\frac{x^3}{\rho_o}}\left(\varepsilon_{o1}+\frac{x^3}{\rho_o}\psi_{o2}^{(1)}+\frac{1}{2}\left(\psi_{o2}\right)^2\right)E \quad (40)$$

so the inner forces are

$$N=\int_A \frac{E\left(x^2,x^3\right)}{1+\frac{x^3}{\rho_o}}\,\mathrm{d}A\,\left(\varepsilon_{o1}+\frac{1}{2}\left(\psi_{o2}\right)^2\right)+\frac{1}{\rho_o}\int_A \frac{x^3 E\left(x^2,x^3\right)}{1+\frac{x^3}{\rho_o}}\,\mathrm{d}A\,\psi_{o2}^{(1)}= \\ =A_{e\rho}\left(\varepsilon_{o1}+\frac{1}{2}\left(\psi_{o2}\right)^2\right)+\frac{Q_{e\rho}}{\rho_o}\psi_{o2}^{(1)}=A_{e\rho}\varepsilon_m+\frac{Q_{e\rho}}{\rho_o}\psi_{o2}^{(1)} \quad (41a)$$

and

$$M=\int_A \frac{x^3 E\left(x^2,x^3\right)}{1+\frac{x^3}{\rho_o}}\,\mathrm{d}A\,\left(\varepsilon_{o1}+\frac{1}{2}\left(\psi_{o2}\right)^2\right)+\frac{1}{\rho_o}\int_A \frac{\left(x^3\right)^2 E\left(x^2,x^3\right)}{1+\frac{x^3}{\rho_o}}\,\mathrm{d}A\,\psi_{o2}^{(1)}= \\ =Q_{e\rho}\left(\varepsilon_{o1}+\frac{1}{2}\left(\psi_{o2}\right)^2\right)+\frac{I_{e\rho}}{\rho_o}\psi_{o2}^{(1)}=Q_{e\rho}\varepsilon_m+\frac{I_{e\rho}}{\rho_o}\psi_{o2}^{(1)}\,. \quad (41b)$$

The ultimate and above all, most accurate model worthy of mentioning assumes that

$$E_{11}=\frac{1}{1+\frac{x^3}{\rho_o}}\left(\varepsilon_{o1}+\frac{x^3}{\rho_o}\psi_{o2}^{(1)}+\frac{1}{2}\left(\psi_{o2}\right)^2\right)+\left(\frac{x^3}{\rho_o}\right)^2\frac{1}{2}\left(\psi_{o2}^{(1)}\right)^2, \\ S_{11}=\frac{1}{1+\frac{x^3}{\rho_o}}\left(\varepsilon_{o1}+\frac{x^3}{\rho_o}\psi_{o2}^{(1)}+\frac{1}{2}\left(\psi_{o2}\right)^2\right)E\left(x^2,x^3\right)+\left(\frac{x^3}{\rho_o}\right)^2\frac{1}{2}\left(\psi_{o2}^{(1)}\right)^2 E\left(x^2,x^3\right) \quad (42)$$

and therefore

$$N=\int_A \frac{E\left(x^2,x^3\right)}{1+\frac{x^3}{\rho_o}}\,\mathrm{d}A\,\left(\varepsilon_{o1}+\frac{1}{2}\left(\psi_{o2}\right)^2\right)+\frac{1}{\rho_o}\int_A \frac{x^3 E\left(x^2,x^3\right)}{1+\frac{x^3}{\rho_o}}\,\mathrm{d}A\,\psi_{o2}^{(1)}+$$

$$+\frac{1}{\rho_o^2}\int_A \left(x^3\right)^2 E\left(x^2,x^3\right)\,\mathrm{d}A\,\frac{1}{2}\left(\psi_{o2}^{(1)}\right)^2 = A_{e\rho}\varepsilon_m + \frac{Q_{e\rho}}{\rho_o}\psi_{o2}^{(1)} + \frac{\mathcal{I}_e}{2\rho_o^2}\left(\psi_{o2}^{(1)}\right)^2 \quad \text{(43a)}$$

$$M = \int_A \frac{x^3 E\left(x^2,x^3\right)}{1+\frac{x^3}{\rho_o}}\,\mathrm{d}A\left(\varepsilon_{o1}+\frac{1}{2}\left(\psi_{o2}\right)^2\right) + \frac{1}{\rho_o}\int_A \frac{\left(x^3\right)^2 E\left(x^2,x^3\right)}{1+\frac{x^3}{\rho_o}}\,\mathrm{d}A\,\psi_{o2}^{(1)} +$$

$$+\frac{1}{\rho_o^2}\int_A \left(x^3\right)^3 E\left(x^2,x^3\right)\,\mathrm{d}A\,\frac{1}{2}\left(\psi_{o2}^{(1)}\right)^2 = Q_{e\rho}\varepsilon_m + \frac{I_{e\rho}}{\rho_o}\psi_{o2}^{(1)} + \frac{\mathcal{P}_e}{2\rho_o^2}\left(\psi_{o2}^{(1)}\right)^2. \quad \text{(43b)}$$

6. SUMMARY

We have derived a new kinematical model, rest on the Kirchhoff hypothesis, for a future stability investigation of circular beams. This model provides the axial strain in a more accurate way than the reviewed literature. We have kept all quadratic terms but the square of the linearized strains. We have provided some possible relations between the strains and the inner forces which might be worthy of considering.

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