DETERMINATION OF DISPLACEMENTS AND STRESSES IN FUNCTIONALLY GRADED HOLLOW SPHERICAL BODIES

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Abstract: The main objective of this paper is the calculation of the thermomechanical stresses and displacements in functionally graded hollow spherical bodies subjected to thermal and mechanical loadings. The material properties are described as power functions of the radial coordinate and the Poisson ratio is constant. It is assumed that the temperature field and the displacement field depend only on the radial coordinate. The analytical solution is derived via the solution for a system of diferential equations which contains a stress function and the displacement field as unknowns.

1. INTRODUCTION

This paper investigates a thermoelastic problem of a functionally graded spherical body. The geometry of the investigated body can be seen in Fig. 1, where the inner radius of the sphere is R_1, the outer radius is denoted by R_2. In order to solve this problem a spherical coordinate system $Or\varphi\vartheta$ is used. First kind thermal boundary conditions are prescribed on the inner and outer spherical surfaces. These constants are non-time dependent given temperature values and they are denoted by t_1 and t_2. It follows that the temperature field t is the function of the radial coordinate r. The uniformly distributed pressure exerted on the inner boundary surface is denoted by p_1, while p_2 is the uniformly distributed pressure which acts on the outer spherical boundary surface.
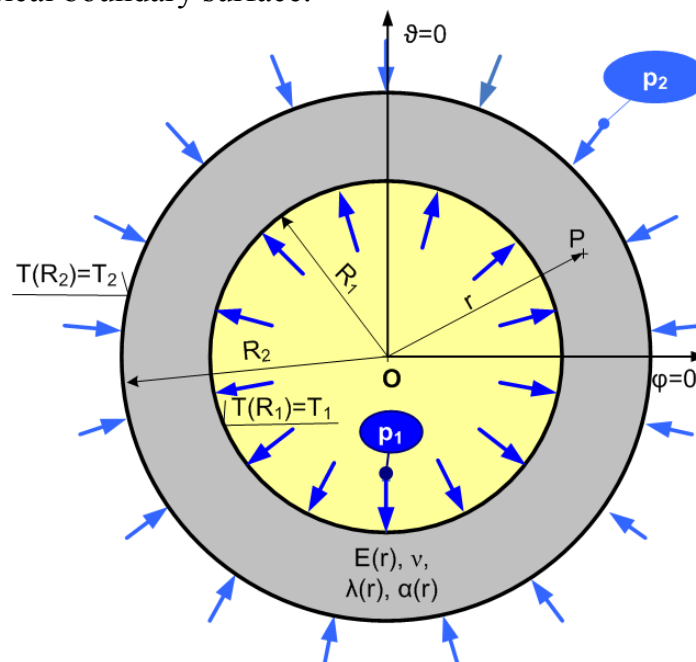

Fig. 1. The functionally graded spherical hollow body.

The material properties are given as

$$E(r) = E_0 r^{m_1}, \quad \alpha = \alpha_0 r^{m_2}, \quad \lambda = \lambda_0 r^{m_3}, \tag{1}$$

where E is the Young modulus, α is the coefficient of linear thermal expansion, λ is the thermal conductivity, r is the radial coordinate, E_0, α_0, λ_0, m_1, m_2 and m_3 are material constants, furthermore the Poisson ratio v is constant.

It is assumed that the radial stresses, the heatflow and the temperature field are all continuous functions of the radial coordinate. Our aim is to determine the displacement field and normal stresses within the spherical component.

2. FORMULATION OF THE PROBLEM

The first step is the calculation of the temperature field when the thermal conductivity is prescribed by Eq. (1). In our problem the first kind thermal boundary conditions are

$$t(R_1) = t_1, \quad t(R_2) = t_2 = 0. \tag{2}$$

In this case the temperature difference field $T(r) = t(r) - t_0$ has the following form [1]:

$$T(r) = T_1 \left(1 - \frac{r^{-m_3-1} - R_1^{-m_3-1}}{R_2^{-m_3-1} - R_1^{-m_3-1}} \right), \quad R_1 \leq r \leq R_2. \tag{3}$$

We note that t_0 denotes the reference temperature at which the body is stress free if there is no deformation and $T_i = t_i - t_0$ $(i = 1, 2)$.

The radial and tangential normal strains ε_r, $\varepsilon_\varphi = \varepsilon_\vartheta$ and the stress-strain relations for a spherical body can be formulated as [2, 3]:

$$\varepsilon_r = \frac{du}{dr}, \quad \varepsilon_\varphi = \varepsilon_\vartheta = \frac{u}{r}, \tag{4}$$

$$\sigma_r = \frac{E}{(1-2v)(1+v)} \left[(1-v)\varepsilon_r + 2v\varepsilon_\varphi - \alpha(1+v)T \right], \tag{5}$$

$$\sigma_\varphi = \sigma_\vartheta = \frac{E}{(1-2v)(1+v)} \left[v\varepsilon_r + \varepsilon_\varphi - \alpha(1+v)T \right], \tag{6}$$

where $u = u(r)$ is the radial displacement field, $\sigma_r(r)$ is the function of radial normal stress and $\sigma_\varphi(r)$ is the tangential stress. The equilibrium equation in the radial direction of the spherical body has the following form [2, 3]:

$$\frac{d\sigma_r}{dr} + \frac{2(\sigma_r - \sigma_\varphi)}{r} = 0, \quad R_1 \le r \le R_2. \tag{7}$$

We reformulate Eq. (7) in the next form

$$\frac{d(r^2 \sigma_r)}{dr} = 2r\sigma_\varphi, \tag{8}$$

therefore the normal stresses can be expressed in terms of the stress function $V=V(r)$ as

$$\sigma_r = \frac{V}{r^2}, \qquad \sigma_\varphi = \frac{1}{2r}\frac{dV}{dr}, \quad R_1 \le r \le R_2. \tag{9}$$

After some manipulations from Eqs. (5-7) and (9) we can derive the next system of ordinary differential equations for the displacement field and the stress function

$$\frac{d}{dr}\begin{bmatrix} u \\ V \end{bmatrix} = \begin{bmatrix} -\dfrac{2\nu}{(1-\nu)}\dfrac{1}{r} & \dfrac{(1-2\nu)(1+\nu)}{(1-\nu)E}\dfrac{1}{r^2} \\ \dfrac{2E}{1-\nu} & \dfrac{2\nu}{1-\nu}\dfrac{1}{r} \end{bmatrix}\begin{bmatrix} u \\ V \end{bmatrix} + \begin{bmatrix} \dfrac{1+\nu}{1-\nu} \\ \dfrac{2E}{1-\nu}r \end{bmatrix}\alpha T. \tag{10}$$

Considering the functions of the material properties given by Eq. (1) the final form for the system of differential equations can be expressed as:

$$\frac{d}{dr}\begin{bmatrix} u \\ V \end{bmatrix} = \begin{bmatrix} -\dfrac{2\nu}{(1-\nu)}\dfrac{1}{r} & \dfrac{(1-2\nu)(1+\nu)}{(1-\nu)E_0}\dfrac{1}{r^{m_1+2}} \\ \dfrac{2E_0 r^{m_1}}{1-\nu} & \dfrac{2\nu}{1-\nu}\dfrac{1}{r} \end{bmatrix}\begin{bmatrix} u \\ V \end{bmatrix} + \begin{bmatrix} \dfrac{1+\nu}{1-\nu} \\ \dfrac{2E_0}{1-\nu}r^{m_1+1} \end{bmatrix}\alpha_0 r^{m_2} T. \tag{11}$$

The general solutions of the radial displacement field and the stress function are power functions of the radial coordinate. The homogeneus solutions are assumed to have the following forms:

$$u_h = C_1 r^\lambda, \qquad V_h = C_2 r^{\lambda+m_1+1}. \tag{12}$$

Applying Eqs. (12) to Eqs. (11) we get the next system of linear equations for the the constants C_1 and C_2

$$\begin{bmatrix} \lambda + \dfrac{2v}{(1-v)} & \dfrac{(1-2v)(1+v)}{(1-v)E_0} \\[3mm] -\dfrac{2E_0}{1-v} & (\lambda + m_1 + 1) - \dfrac{2v}{1-v} \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{13}$$

From the solutions for the previously presented system of equations it follows that

$$u_h = \frac{(\lambda + m_1 + 1)(1-v) - 2v}{2E_0} C_1 r^{\lambda_1} + \frac{(\lambda + m_1 + 1)(1-v) - 2v}{2E_0} C_2 r^{\lambda_2}, \tag{14}$$

$$V_h = C_1 r^{\lambda_1 + m_1 + 1} + C_2 r^{\lambda_2 + m_1 + 1}, \tag{15}$$

$$\lambda_{1,2} = \frac{-1 - m_1 \pm \sqrt{(m_1 + 1)^2 - 4A}}{2}, \quad A = \frac{2\left[v(m_1 + 1) - 1\right]}{1-v}, \tag{16}$$

$$C_1 = \frac{(\lambda + m_1 + 1)(1-v) - 2v}{2E_0} C_2. \tag{17}$$

The following notations will be used for the computation of the particular solutions:

$$t_1 = T_1 \left(1 + \frac{R_1^{-m_3 - 1}}{R_2^{-m_3 - 1} - R_1^{-m_3 - 1}}\right) \frac{\alpha_0}{1-v}, \quad t_2 = \frac{T_1}{R_2^{-m_3 - 1} - R_1^{-m_3 - 1}} \frac{\alpha_0}{1-v} \tag{18}$$

The first particular solution is obtained the next system of differential equations:

$$\frac{d}{dr}\begin{bmatrix} u \\ V \end{bmatrix} = \begin{bmatrix} -\dfrac{2v}{(1-v)}\dfrac{1}{r} & \dfrac{(1-2v)(1+v)}{(1-v)E_0}\dfrac{1}{r^{m_1+2}} \\[3mm] \dfrac{2E_0 r^{m_1}}{1-v} & \dfrac{2v}{1-v}\dfrac{1}{r} \end{bmatrix} \begin{bmatrix} u \\ V \end{bmatrix} + \begin{bmatrix} 1+v \\ 2E_0 r^{m_1+1} \end{bmatrix} t_1 r^{m_2}, \tag{19}$$

$$u_{p1} = D_1 r^{m_2 + 1}, \qquad V_{p1} = D_2 r^{m_2 + m_1 + 2}, \tag{20}$$

and we have

$$D_1 = \frac{-(1+v)\left(m_1 + m_2 + 2 - \dfrac{2v}{1-v}\right) + \dfrac{2(1+v)(1-2v)}{1-v}}{\left(m_2 + 1 + \dfrac{2v}{1-v}\right)\left(m_1 + m_2 + 2 - \dfrac{2v}{1-v}\right) - \dfrac{2(1+v)(1-2v)}{(1-v)^2}} t_1, \tag{21}$$

$$D_2 = \frac{-2E_0\left(m_1 - m_3 - \dfrac{2v}{1-v}\right) + \dfrac{2(1+v)E_0}{1-v}}{\left(m_2 + 1 + \dfrac{2v}{1-v}\right)\left(m_1 + m_2 + 2 - \dfrac{2v}{1-v}\right) - \dfrac{2(1+v)(1-2v)}{(1-v)^2}} t_1. \qquad (22)$$

The remaining particular solutions can be represented as

$$u_{p2} = F_1 r^{m_2 - m_3}, \qquad V_{p2} = F_2 r^{m_2 + m_1 - m_3 + 1}, \qquad (23)$$

$$F_1 = \frac{(1+v)\left(m_1 + m_2 - m_3 + 1 - \dfrac{2v}{1-v}\right) - \dfrac{2(1+v)(1-2v)}{1-v}}{\left(m_2 - m_3 + \dfrac{2v}{1-v}\right)\left(m_1 + m_2 - m_3 + 1 - \dfrac{2v}{1-v}\right) - \dfrac{2(1+v)(1-2v)}{(1-v)^2}} t_2, \qquad (24)$$

$$F_2 = \frac{2E_0\left(m_2 + 1 + \dfrac{2v}{1-v}\right) - \dfrac{2(1+v)E_0}{1-v}}{\left(m_2 + 1 + \dfrac{2v}{1-v}\right)\left(m_1 + m_2 + 2 - \dfrac{2v}{1-v}\right) - \dfrac{2(1+v)(1-2v)}{(1-v)^2}} t_1. \qquad (25)$$

The summarized form of the general solution for the displacement field and the stress function are as follows

$$u = \frac{(\lambda_1 + m_1 + 1)(1-v) - 2v}{2E_0} C_1 r^{\lambda_1} + \frac{(\lambda_2 + m_1 + 1)(1-v) - 2v}{2E_0} C_2 r^{\lambda_2} + \\ + D_1 r^{m_2 + 1} + F_1 r^{m_2 - m_3}, \qquad (26)$$

$$V = C_1 r^{\lambda_1 + m_1 + 1} + C_2 r^{\lambda_2 + m_1 + 1} + D_2 r^{m_2 + m_1 + 2} + F_2 r^{m_2 + m_1 - m_3 + 1}. \qquad (27)$$

In order to determine the unknown C_1 and C_2 constants the next stress boundary conditions will be used:

$$\frac{V(R_1)}{R_1^2} = -p_1, \qquad \frac{V(R_2)}{R_2^2} = -p_2. \qquad (28)$$

4. NUMERICAL EXAMPLES

For the numerical example the following data are used:

$$R_1 = 1\,\text{m};\ R_2 = 1.15\,\text{m};\ E_0 = 210\,\text{GPa};\ m_1 = m_2 = m_3 = m;\ \nu = 0.3;$$

$$\lambda = 48\frac{\text{W}}{\text{mK}};\alpha_0 = 1.2 \cdot 10^{-6}\,\frac{1}{\text{K}};p_1 = 20\,\text{MPa};p_2 = 0\,\text{MPa};T_1 = 573\,\text{K};T_2 = 273\,\text{K}$$

$$\alpha_2 = 12 \cdot 10^{-6}\,\frac{1}{\text{K}};p_i = 20\,\text{MPa};p_o = 0\,\text{MPa};t_1 = 523\,\text{K};t_4 = 273\,\text{K}.$$

Figs. 4-6 indicate the results of this problem, which was solved by *Maple 15*, for three cases $(m = 0.1,1,3)$.
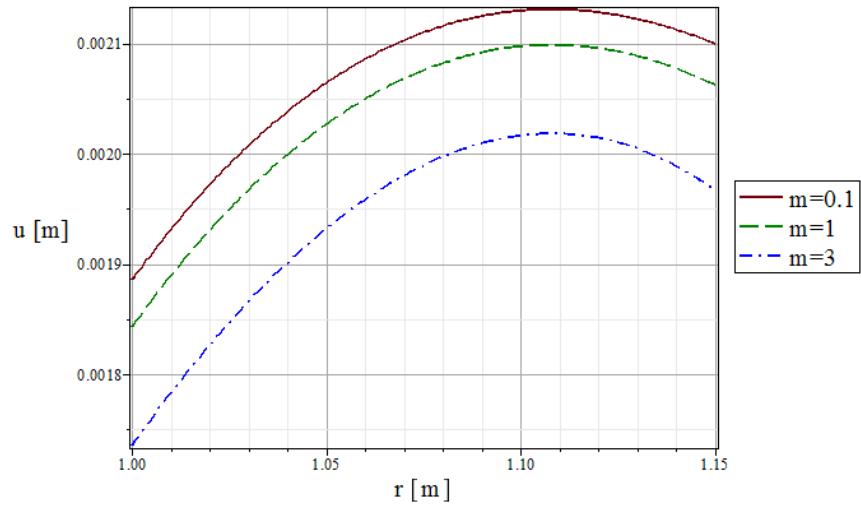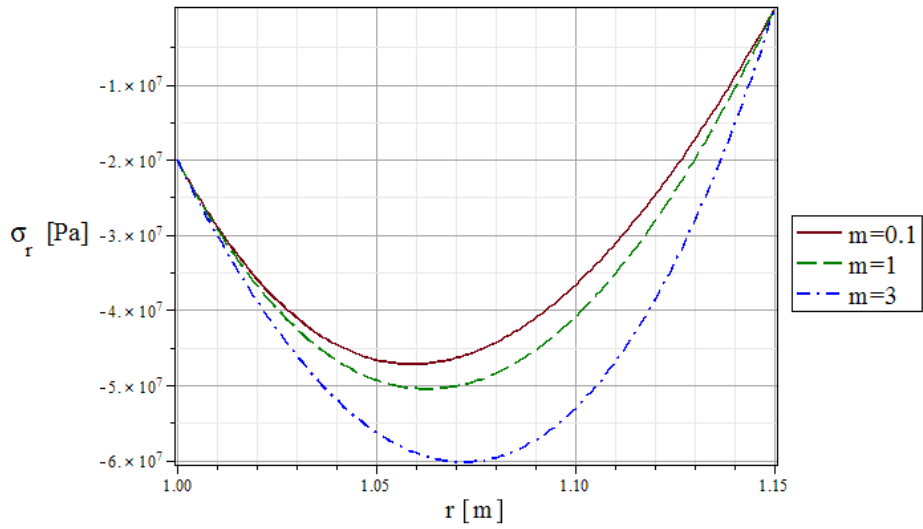

Figure 4. The radial displacement fields.

Figure 5. The radial normal stresses.

In Figs. 4-6 the red solid lines are the results of the thermomechanical problem with *m=0.1*, the blue lines indicate the results for the case when *m=3*. The green dash lines illustrate the functions when *m=1*.
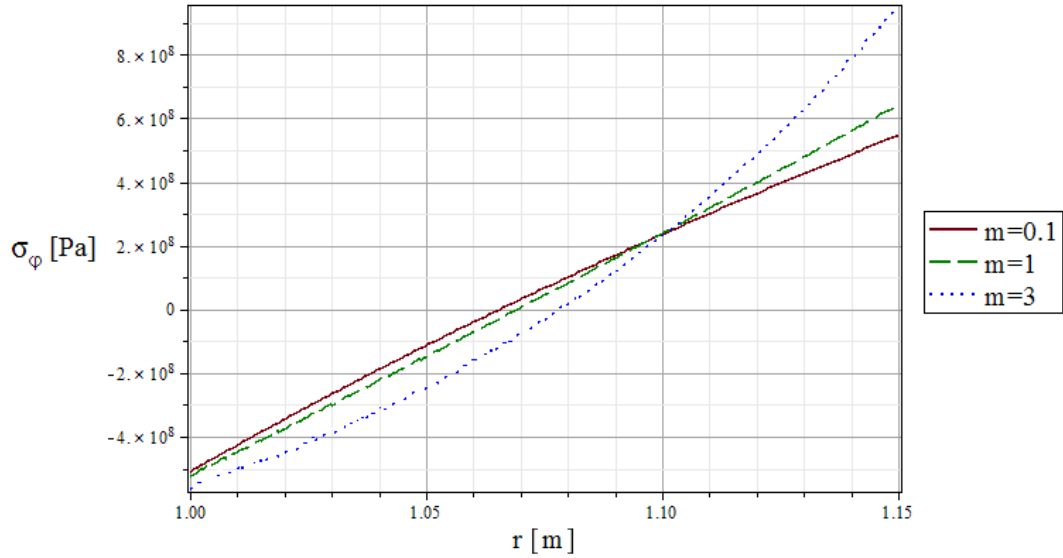

Figure 6. The tangential normal stresses of the three cases.

5. CONCLUSIONS

The main objective of this paper was to present an analytical solution for the displacement field and the associated stresses in functionally graded spherical bodies subjected to mechanical and thermal loads. To solve this problem a system of ordinary diferential equations is derived which contains a stress function and the displacement field as unknowns. The developed solution can be utilized as Benchmark solutions for numerical methods to verify the accuracy of the considered numerical methods.

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