# BUCKLING OF ANNULAR PLATES WITH SHELL-STIFFENING AND ELASTIC RESTRAINS ON THE BOUNDARY

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## 1. INTRODUCTION

A number of papers have been devoted to the stability problem of circular plates though only a few have dealt with the influence of stiffening.

There are various methods for increasing the resistance of a circular plate to buckling. For example, one can apply an internal ring support, which can be either rigid or elastic. Thevendran and Wang have examined the buckling problem of annular plates which are simply supported with elastic rotational restrains at the inner or outer boundary [1]. Laura et al. have investigated the buckling of circular, solid and annular plates with an intermediate circular support under the assumption of axisymmetric deformations [2]. By the use of the Kirchhoff-Love plate theory [3] and the Mindlin–Reissner theory [4] Wang and his co-authors studied the same structure under the assumption of non-axisymmetric deformations. Rao and Rao have analysed the buckling of circular plates which are supported along concentric rings. The supports applied are simple or translational and/or torsional elastic restrains [5, 6]. The authors have also investigated a circular plate with elastic foundation [7].

We can also use discrete stiffeners, which are applied to the plates. The effect on stability of a ring stiffener on the boundary of a circular plate has been investigated by Phillips and Carney [8]. Rossettos and Miller have investigated symmetric and asymmetric buckling of a circular plate which is stiffened by a ring at an internal radius [9, 10]. The axial rigidity of the stiffening ring has been ignored. Frostig and Simitses have examined a similar structure but they have not used the simplifications of the aforementioned article [11, 12]. The stiffening ring is modeled as a curved beam.

Szilassy dealt with the stability of circular and annular plates stiffened by a cylindrical shell on its outer boundary in his PhD. thesis [13] and in a further article [14]. It was assumed that (i) the load is an in-plane axisymmetric dead one and (ii) the deformations of the annular plate and the cylindrical shell are also axisymmetric.

The present paper deals with the axisymmetric and non axisymmetric buckling of annular plates which are stiffened by a cylindrical shell on the outer boundary, and an elastic restrain against torsion is attached on the inner boundary. The paper outlines the basic assumptions, the governing equations as well as the boundary and continuity conditions. Numerical results are also shown. These represent the influence of shell geometry on buckling load. As regards the cylindrical shell we shall utilize some results from Vlasov [15].

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## 2. PROBLEM FORMULATION

We shall examine the buckling of the structure shown in Fig. 1. The structure is subjected to a constant radial load in the middle plane of the plate. A circular shell is attached on the outer boundary while a torsional spring support is applied along the inner boundary.

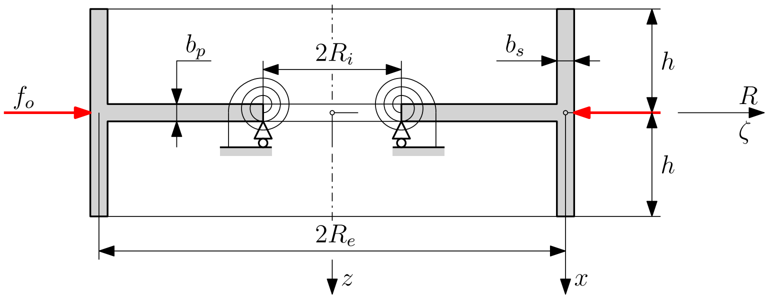


Figure 1. The structure and its load

We shall assume that the plate and the shell are thin, consequently we can apply the Kirchhoff theory of plates and shells. It is a further assumption that the shell and plate are made of the same isotropic material for which $E$ and $\nu$ are the Young-modulus and the Poisson ratio, respectively. Heat effects are not taken into account.

Under the assumption of small, non-axisymmetric and linearly elastic deflection we shall determine (a) the critical buckling load of the structure and (b) how does the shell stiffening affect the critical load.

The deformation of the structural elements is analysed separately in the cylindrical coordinate system $(R, \varphi, z)$ used for the equations of the plate and in the coordinate system $(\zeta, \varphi, x)$ for the cylindrical shell. The coordinate systems are shown in Fig. 2.

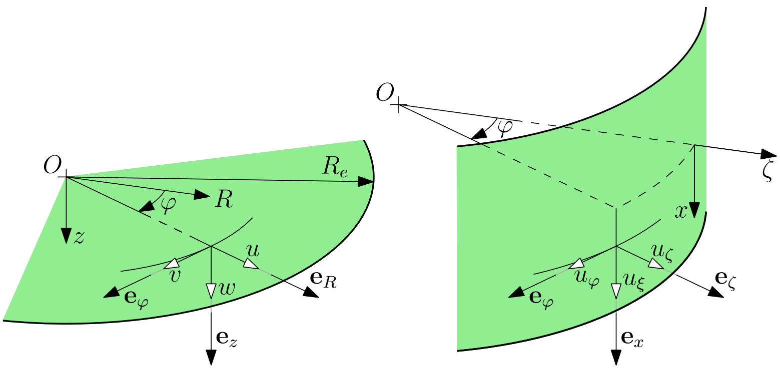


Figure 2. Coordinate systems

## 3. EQUATIONS FOR THE PLATE

We separate the structural elements in order to solve the problem. Fig 3. shows the plate and the cylindrical shell together with the in-plane forces $f_o$ and $f$ which are acting between these elements.

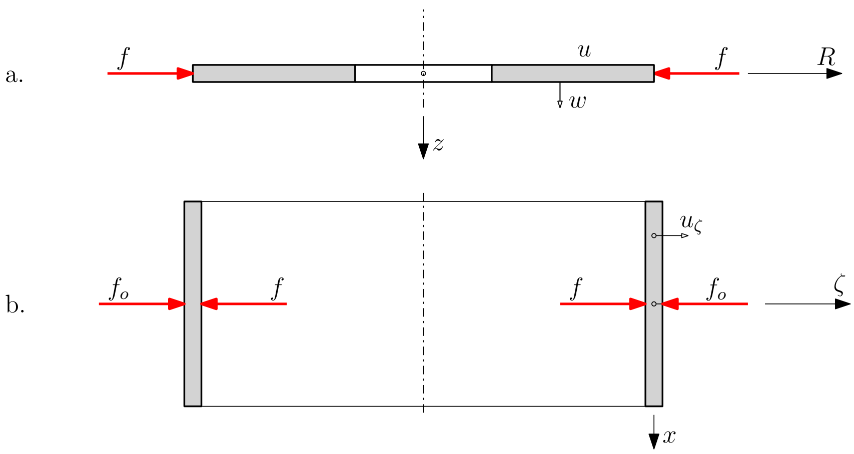


Figure 3. Free body diagram for plate and shell

The inner forces $N_R$, $N_\varphi$, $N_{R\varphi}$ in the plate are axisymmetric because the in-plane load exerted on the outer boundary is axisymmetric as well. Introducing the dimensionless coordinate $\rho = \frac{R}{R_e}$, the inner forces take the form

$$N_R = -A + \frac{B}{\rho^2}\,, \qquad N_\varphi = -A - \frac{B}{\rho^2}\,, \qquad N_{R\varphi} = 0\,. \tag{1}$$

The constants $A$ and $B$ depend on the boundary conditions. For an annular plate with free outer boundary we obtain

$$A = f\frac{1}{1-\rho_i^2}\,, \qquad B = f\frac{\rho_i^2}{1-\rho_i^2} \tag{2}$$

respectively, where $\rho_i = R_i/R_e$. The radial displacement is given by the formula

$$u = \frac{\rho R_e}{b_p E}\left[-A\,(1-\nu) - \frac{B}{\rho^2}\,(1+\nu)\right]\,. \tag{3}$$

The deflection $w$ of the plate should fulfill the differential equation

$$I_1 E_1 \tilde{\triangle}\tilde{\triangle} w - \left[N_R \frac{\partial^2 w}{\partial \rho^2} + 2N_{R\varphi}\frac{\partial}{\partial \rho}\left(\frac{1}{\rho}\frac{\partial w}{\partial \varphi}\right) + N_\varphi\left(\frac{1}{\rho}\frac{\partial w}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 w}{\partial \varphi^2}\right)\right] = 0\,, \tag{4}$$

where

$$\tilde{\triangle} = \frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial \varphi^2}\,, \qquad I_1 = \frac{b_p^3}{12}\,, \qquad E_1 = \frac{E}{1-\nu^2}\,, \tag{5}$$

and $b_p$ is the thickness of the plate. We expand the solution for $w$ in a Fourier series of the form

$$w = w_o + \sum_{m=0}^{1}\sum_{n=1}^{\infty} \overset{m}{w}_n(\rho)\cos\left(n\varphi - m\frac{\pi}{2}\right) \tag{6}$$

and substitute it into (4). We obtain that the amplitudes $w_o(\rho)$ and $\overset{m}{w}_n(\rho)$ should fulfill the following differential equations:

$$\left(\frac{\mathrm{d}^4}{\mathrm{d}\rho^4} + \frac{2}{\rho}\frac{\mathrm{d}^3}{\mathrm{d}\rho^3} - \frac{1+2n^2}{\rho^2}\frac{\mathrm{d}^2}{\mathrm{d}\rho^2} + \frac{1+2n^2}{\rho^3}\frac{\mathrm{d}}{\mathrm{d}\rho} + \frac{n^4-4n^2}{\rho^4}\right)\overset{m}{w}_n -$$
$$-\left(N_R\frac{R_k^2}{I_1E_1}\frac{\partial^2}{\partial\rho^2} + N_\varphi\frac{R_k^2}{I_1E_1}\left[\frac{1}{\rho}\frac{\partial}{\partial\rho} - \frac{n^2}{\rho^2}\right]\right)\overset{m}{w}_n = 0\,; \quad m = 0, 1, \quad n = 0, 1, 2, \ldots \tag{7}$$

The solutions of the equations are sought by an appropriate numerical method.

The rotation $\psi_\varphi$, the bending moment $M_R$ and the shear force $Q_R$ are given by the relations

$$\vartheta = -\frac{1}{R_e}\frac{\mathrm{d}w}{\mathrm{d}\rho}\,, \qquad M_R = -\frac{I_{1p}E_{1p}}{R_e^2}\left[\frac{\partial^2 w}{\partial \rho^2} + \frac{\nu_p}{\rho}\left(\frac{\partial w}{\partial \rho} + \frac{1}{\rho}\frac{\partial^2 w}{\partial \varphi^2}\right)\right]\,, \tag{8a}$$

$$Q_R = I_{1p}E_{1p}\frac{1}{R_e^3}\frac{\partial}{\partial \rho}\left(\tilde{\triangle} w\right) - \frac{N_R}{R_e}\frac{\partial w}{\partial \rho}\,. \tag{8b}$$

These physical quantities can be written in the same form as the series (6). It is obvious that the amplitude functions of $\psi_\varphi$, $M_R$ and $Q_R$ can all be given in terms of the $\overset{m}{w}_n$ amplitudes of $w$ – the details are omitted here.

## 4. Equations for the cylindrical shell

Assuming that the shell is subjected to a radial load, the fundamental equations set up for the displacement coordinates $u_\xi$, $u_\varphi$ and $u_\zeta$ will be fulfilled identically if we calculate the displacement coordinates from the Galerkin function $\phi$ using the relations [16]

$$u_\xi = \frac{\partial^3 \phi}{\partial \xi \partial \varphi^2} - \nu\frac{\partial^3 \phi}{\partial \xi^3}\,, \quad u_\varphi = -\frac{\partial^3 \phi}{\partial \varphi^3} - (2+\nu)\frac{\partial^3 \phi}{\partial \xi^2 \partial \varphi}\,, \quad u_\zeta = \nabla^2\nabla^2\phi\,, \tag{9}$$

where $\phi$ should satisfy the following DE (the distributed load $p_z = 0!$):

$$\nabla^2\nabla^2\nabla^2\nabla^2\phi + 4\beta^2\frac{\partial^4 \phi}{\partial \xi^4} = \frac{4\beta^4 R_s^2}{Eb_s}p_z = 0\,,\ \nabla^2 = \frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \varphi^2}\,,\ \beta^4 = 3\left(1-\nu^2\right)\frac{R_s^2}{b_s^2}\,. \tag{10}$$

The physical quantities in the shell set up for the displacement field can also be given in terms of the Galerkin function. The following relations present those which appear in the boundary and continuity conditions:

$$\psi_\varphi = \frac{1}{R_e}\frac{\partial}{\partial \xi}\nabla^2\nabla^2\phi\,, \qquad N_{xx} = \frac{Eb_s}{R_e}\frac{\partial^4 \phi}{\partial \xi^2 \partial \varphi^2}\,, \qquad N_{\varphi x} = -\frac{Eb_s}{R_e}\frac{\partial^4 \phi}{\partial \xi^3 \partial \varphi}\,, \tag{11a}$$

$$M_{xx} = -\frac{Eb_s}{4\hat{\beta}^4}\left[\frac{\partial^2}{\partial \xi^2} + \nu_s\frac{\partial^2}{\partial \varphi^2}\right]\nabla^2\nabla^2\phi\,, \quad M_{x\varphi} = -\frac{Eb_s\left(1-\nu_s\right)}{4\hat{\beta}^4}\frac{\partial^2}{\partial \xi \partial \varphi}\nabla^2\nabla^2\phi\,, \tag{11b}$$

$$Q_{x\zeta} = \frac{1}{R_e}\frac{Eb_s}{4\hat{\beta}^4}\frac{\partial}{\partial \xi}\nabla^2\nabla^2\nabla^2\phi\,, \tag{11c}$$

(11d)

Similarly to equation (6) we also assume that $\phi$ is expanded in a Fourier series:

$$\phi\left(\xi,\varphi\right) = \phi_o\left(\xi\right) + \sum_{m=0}^{1}\sum_{n=1}^{\infty}\overset{m}{\phi}_n\left(\xi\right)\cos\left(n\varphi - m\frac{\pi}{2}\right)\,. \tag{12}$$

After substituting (12) into (10) we obtain the differential equations for the Fourier coefficients. It can also be shown that the real solution for them take the form

$$\overset{m}{\phi}_n = \sum_{k=1}^{2}\left[\overset{m}{K}_{nk}\sinh\left(\beta_{nk}\xi\right)\sin\left(\alpha_{nk}\xi\right) + \overset{m}{M}_{nk}\sinh\left(\beta_{nk}\right)\cos\left(\alpha_{nk}\xi\right) + \right.$$
$$\left. + \overset{m}{P}_{nk}\cosh\left(\beta_{nk}\xi\right)\sin\left(\alpha_{nk}\xi\right) + \overset{m}{S}_{nk}\cosh\left(\beta_{nk}\xi\right)\cos\left(\alpha_{nk}\xi\right)\right]\,, \tag{13}$$

where $\alpha_{nk}$ and $\beta_{nk}$ are some characteristic values obtained from the characteristic polynomial of the aforementioned equations. The quantities $\overset{m}{M}_{nk}, \ldots, \overset{m}{J}_{nk}$ constitute altogether eight integration constants.

Every physical quantity in the shell can be written in a form similar to that of Eq. (6) – we should write $\xi$ instead of $\rho$ there. The coefficients of these can be given in terms of $\phi_o$ and $\overset{m}{\phi}_n$.

## 5. BOUNDARY- AND CONTINUITY CONDTIONS

A solution for the amplitude of the displacement field on the middle surface of the plate contains four, while a solution for $\overset{m}{\phi}_n$ involves eight integration constants. The shell is divided in two separate parts on the intersection line of the structural elements. Therefore we need two solutions, one for each shell part, consequently we have to determine altogether 20 integration constants.

The radial displacement is axisymmetric, i.e., $\overset{m}{u}_{\zeta\, n}(\xi = 0) = 0$ if $n \neq 0$. Consequently we cannot prescribe any condition for the shear force $\overset{m}{Q}_{xz\, n}$.

Since the plane stress problem is axisymmetric, $\overset{m}{v}_n(\rho = 1) = \overset{m}{u}_{\varphi\, n}(\xi = 0) = 0$ as well. Consequently, we cannot prescribe continuity conditions for the inner forces $\overset{m}{N}_{R\varphi\, n}$ and $\overset{m}{N}_{x\varphi\, n}$. However, the axisymmetric parts of these quantities are zero.

The shell and plate deform together on the intersection line of the middle surfaces of the shell and the plate, so it is clear that the following kinematic continuity conditions should also be fulfilled:

$$u_\xi(\xi = 0) = -w(\rho = 1) \ , \qquad \vartheta(\xi = 0) = \psi_\varphi(\rho = 1) \ . \tag{14}$$

It follows from the global equilibrium of the structure that the axisymmetric part of the shear force should meet the condition $Q_{Ro} = 0$. Otherwise the continuity condition

$$Q_R(\rho = 1) - N_{xx}(\xi = +0) + N_{xx}(\xi = -0) = 0 \tag{15a}$$

should be fulfilled. As regards the bending moments equation

$$M_R(\rho = 1) - M_{xx}(\xi = +0) + M_{xx}(\xi = -0) = 0 \tag{15b}$$

is the continuity condition.

The inner boundary of the plate is elastically restrained, and deflection is restrained. The boundary conditions therefore are

$$w(\rho = \rho_i) = 0 \ , \qquad M_R(\rho = \rho_i) = k\vartheta(\rho = \rho_i) \ . \tag{16}$$

where $k$ is the spring constant of the torsional spring. In the computation we use the dimensionless $\mathfrak{K} = \frac{kR_e}{I_1 E_1}$.

Since the boundaries of the shell with coordinates $\xi = \pm h/R_e$ are free, the following boundary conditions should be satisfied:

$$N_{xx}(\xi = \pm h/R_e) = 0 \, , \qquad N_{x\varphi} + M_{x\varphi}/R_e(\xi = \pm h/R_e) = 0 \ , \tag{17a}$$

$$M_{xx}(\xi = \pm h/R_e) = 0 \, , \qquad Q_{xz} - \frac{1}{R_e}\frac{\mathrm{d}}{\mathrm{d}\varphi}M_{x\varphi}(\xi = \pm h/R_e) = 0 \ . \tag{17b}$$

The boundary- and continuity conditions provide twenty homogenous algebraic equations for the 20 integration constants. These equations involve $f_o$ as a parameter. The critical value of $f_o$ can be determined from the condition that the system determinant should vanish.

## 6. NUMERICAL RESULTS

The computational results for the problem we have established above are presented in the following figures. The graphs show the critical loads $\mathfrak{F}_o$ of the structure against the height of the shell.

We can observe the influence of the spring constant in Figure 4. In the two limiting cases (if $\mathfrak{K} = 0$ and $\mathfrak{K} \to \infty$) the results are the same as for simply supported and clamped inner boundary.

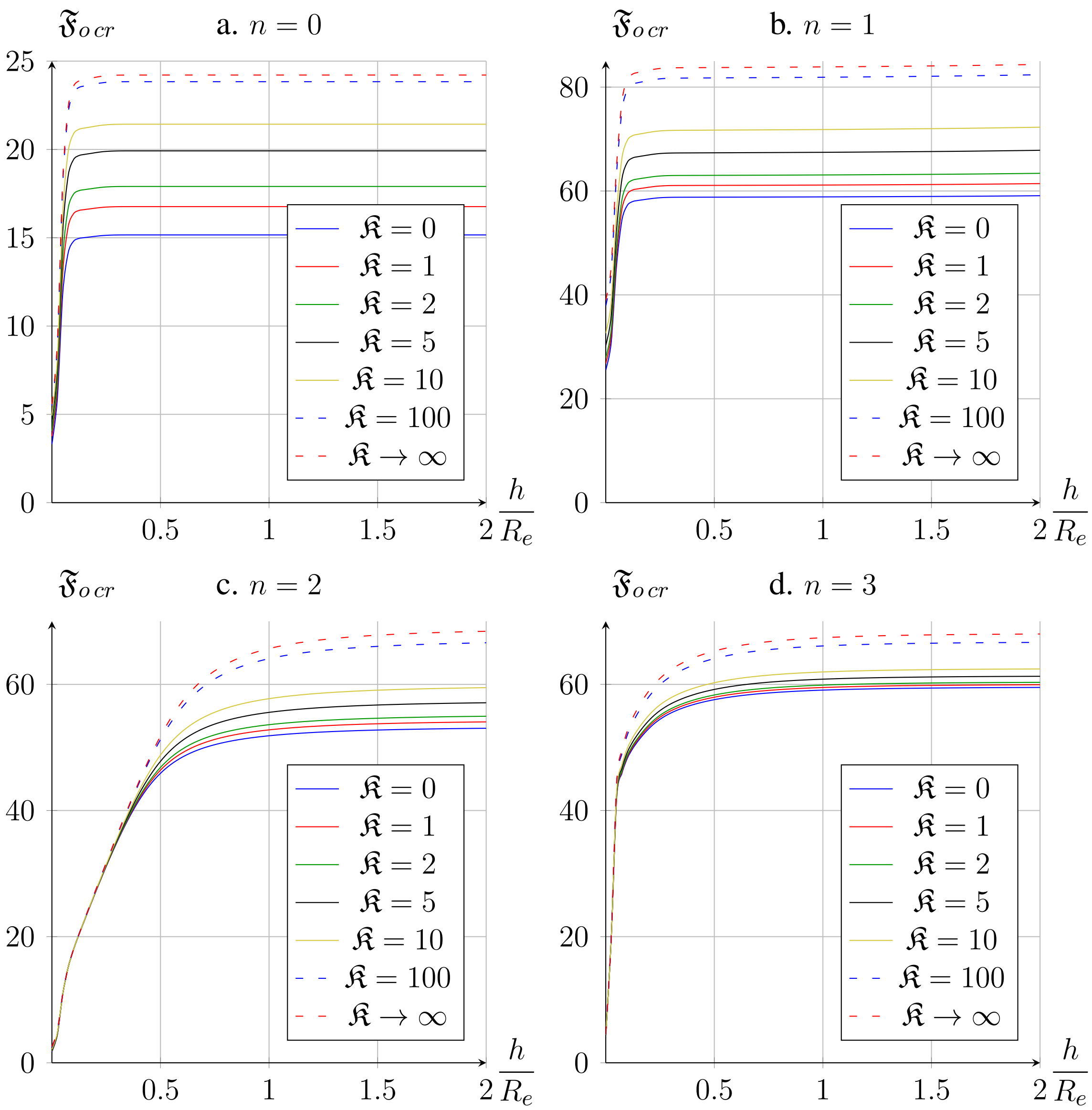


Figure 4. Critical loads for various spring constants, $n = 0, \ldots, 3$

In Figure 5. the influence of the inner radius is presented. We can notice that the critical load belongs to non-axisymmetric deformations if the shell height is below a certain limit. This limit increases if the inner radius is getting larger.

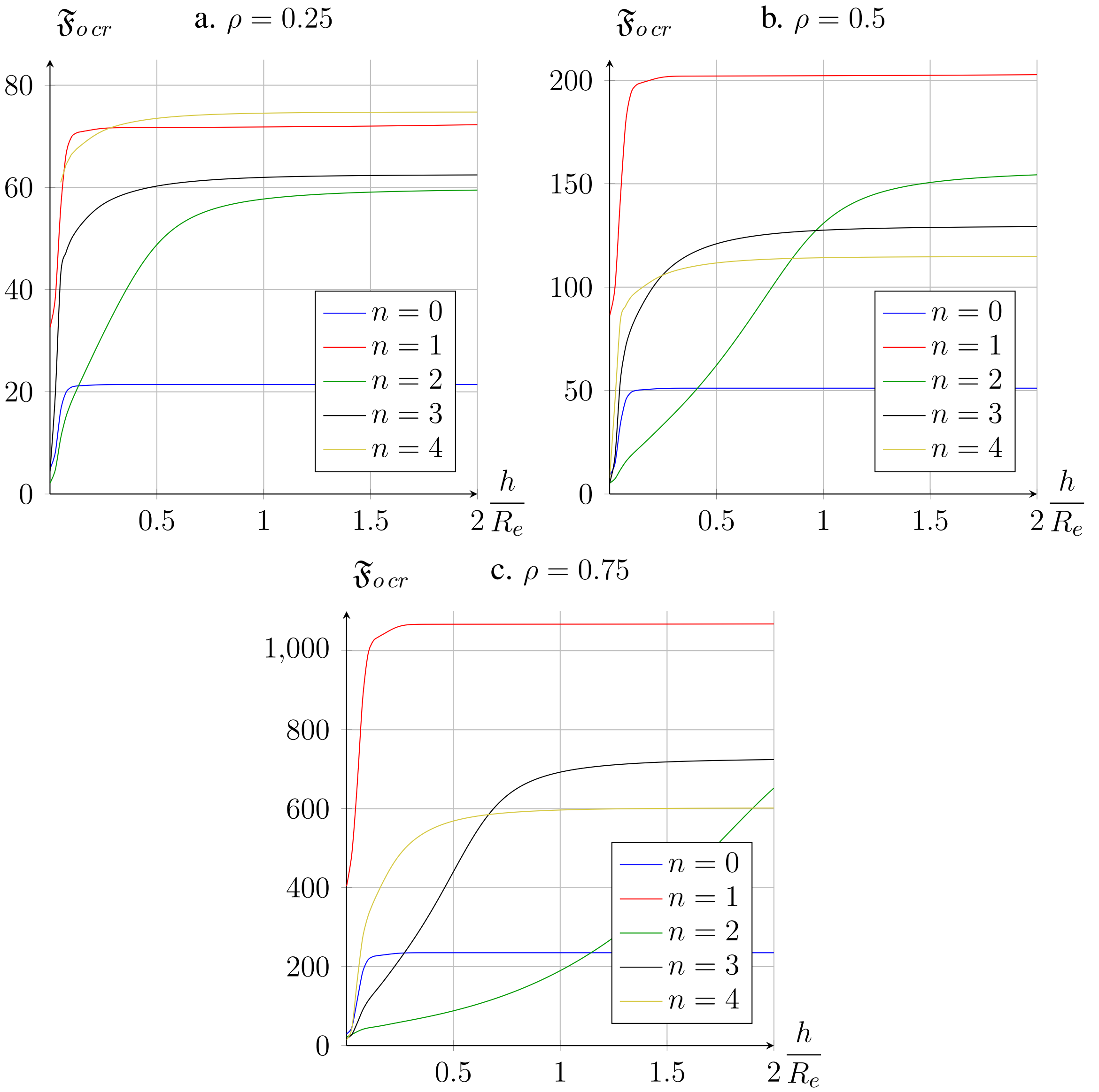


Figure 5. Critical loads for various values of $n$, $\mathfrak{K} = 10$, $\rho = 0.25,\ 0.5,\ 0.75$

## 7. CONCLUDING REMARKS

The present paper has established the equations that can be used to determine the critical load of annular plates stiffened by a cylindrical shell and elastically supported under the assumption of axisymmetric and non-axisymmetric deformations. We have clarified what the continuity conditions are between the two separate elements of the structure. We have also presented the solutions for the critical load of the solid circular plate assuming axisymmetric and non-axisymmetric deformations. It is obvious from the results that the stiffening significantly increases the critical load.

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