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EVOLUTION OF A GENERALIZED POPULATION MODEL

István Fazekas[1], ***Attila Perecsényi***[2]

[1]D.Sc., Full Professor , [2]PhD student
[1,2]*Faculty of Informatics, University of Debrecen*

1 INTRODUCTION

Nowadays there are many random graph models to describe the real networks properties. It is well known that the real networks have scale-free property i.e. they have power-law degree distribution, that is $p_k \sim Ck^{-\gamma}$ as $k \to \infty$, where p_k denotes the relative frequency of vertices with degree k.

In [1] Barabási and Albert introduced a preferential attachment rule. In their model at each step a new vertex is born and it is attach to m old vertices according to the degrees of vertices. It means that a vertex with degree d_k is chosen with probability $d_k / \sum_h d_h$.

Ostroumova, Ryabchenko and Samosvat in [5] defined a general class of preferential attachment type random graph models and they showed results, including the power-law degree distribution.

In Section 2 we study a general class of population models which includes the above mentioned random graph model of Ostroumova, Ryabchenko and Samosvat. After the class is defined precisely, we show an asymptotic theorem.

At the end of this paper we apply our theorem to the weights of cliques in N-interaction model (see [4]), and obtain their power-law distribution.

2 MODEL S AND ITS ASYMPTOTIC BEHAVIOUR

We describe the evolution of a population. The evolution procedure defined below till equation (2.3) is called model S. The evolution starts at time 0 with maximum t individuals. At each time $n = 1, 2, \ldots$ maximum t individuals are born. Any individual is characterized by its score. At birth the score is u with high probability. More precisely the score of a new individual is at least u and at time n

$$\mathbb{P}\,(\text{the score of a new individual} > u) = \mathrm{O}\left(\frac{1}{n}\right). \tag{2.1}$$

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The score of the individual i at time n is denoted by $S_n(i)$. Let $\mathcal{F}_n$ denote the past of the population up to time n. The evolution of the score is described by the following equations

$$\mathbb{P}\left(S_{n+1}(i) = S_n(i) + 1 | \mathcal{F}_n\right) = a\frac{S_n(i)}{n} + b\frac{1}{n} + \mathrm{O}\left(\left(\frac{S_n(i)}{n}\right)^2\right),$$

$$\mathbb{P}\left(S_{n+1}(i) = S_n(i) | \mathcal{F}_n\right) = 1 - a\frac{S_n(i)}{n} - b\frac{1}{n} + \mathrm{O}\left(\left(\frac{S_n(i)}{n}\right)^2\right), \quad (2.2)$$

$$\mathbb{P}\left(S_{n+1}(i) > S_n(i) + 1 | \mathcal{F}_n\right) = \mathrm{O}\left(\left(\frac{S_n(i)}{n}\right)^2\right),$$

where a and b are fixed non-negative numbers. So at each step the score is increased by 1 or 0, the higher increasing is of low probability. Assume also that the total increase of the scores is at most t at each step. Denote by ξ_n the number of new individuals at time n. Assume that

$$\mathbb{E}\xi_n = m + \mathrm{O}\left(\frac{1}{n}\right) \quad (2.3)$$

where $m > 0$.

Let $X_n(s)$ denote the number of individuals having score s at time n. Let $\Theta(x)$ denote a quantity with $|\Theta(x)| < x$. The first theorem shows that the expectation of the score distribution is scale-free.

Theorem 2.1. *Suppose that the conditions of Model S are satisfied. Let* $a > 0$. *Then for any fixed* $s = u, u+1, u+2, \dots$

$$\mathbb{E}X_n(s) = c(u,s)\left(n + \Theta\left(K s^{2+\frac{1}{a}}\right)\right) \quad (2.4)$$

for all n, *where* K *is a fixed finite constant,*

$$c(u,s) = \frac{\Gamma\left(s + \frac{b}{a}\right)\Gamma\left(u + \frac{b+1}{a}\right)}{a\Gamma\left(s + \frac{b+a+1}{a}\right)\Gamma\left(u + \frac{b}{a}\right)} m \quad (2.5)$$

and Γ *denotes the* Γ*-function. Moreover*

$$c(u,s) \sim \frac{m\Gamma\left(u + \frac{b+1}{a}\right)}{a\Gamma\left(u + \frac{b}{a}\right)} s^{-1-\frac{1}{a}} \quad as \quad s \to \infty. \quad (2.6)$$

Proof. The proof follows the same method as used in [5].

3 AN APPLICATION

In this section we apply Theorem 2.1 to the N-interactions random graph model (see [4]). First we recall that, a complete graph with M vertices we call an M-clique. Let $N \geq 3$, $0 < p \leq 1$, $0 \leq r \leq 1$ and $0 \leq q \leq 1$ are fixed numbers. We start at time 0 with an N-clique. Every N-clique and every M-subclique $(0 < M < N)$ have an initial weight 1. The weight of non-existing clique is considered to be 0. In every time step N vertices interact each other, and we draw all non-existing edges between them, so we obtain an N complete graph. The weights of all subcliques are increased by 1.

At each step we have two options. On the one hand, with probability p, a new vertex is born and interact with $N-1$ old vertices. On the other hand, with probability $(1-p)$, we do not add any new vertex and N old vertices interact with each other.

When a new vertex is born there are two possibilities. With probability r we choose an $(N-1)$-clique according to the weights of the $(N-1)$-cliques, based on preferential attachment, or with probability $(1-r)$ we choose $N-1$ old vertices uniformly. The $N-1$ old vertices chosen interact with the new vertex. That is they form a new N-clique. The other case, when we do not add new vertex, there are two possibilities again. With probability q we choose an N-clique according to the weights of the N-cliques, based on preferential attachment, or with probability $(1-q)$ we choose N old vertices uniformly. The N old vertices chosen will form an N-clique.

In any step and in any case the weight of the N-clique constructed in that step and the weights of its subcliques are increased by 1.

Corollary 3.1. Assume that either $r > 0$ or $(1-p)q > 0$. Let $X_n(s)$ denote the number of M-cliques having weight s at time n where $M > 1$, and let

$$a = pr\frac{N-M}{N} + (1-p)q,$$
$$b = 0,$$
$$m = p\binom{N-1}{M-1} + p(1-r)\binom{N-1}{M} + (1-p)(1-q)\binom{N}{M},$$
$$u = 1$$

in Theorem 2.1.

Then we obtain

$$\frac{\mathbb{E}X_n(s)}{n} \to c(u,s) = \frac{\Gamma(s)\Gamma\left(1+\frac{1}{a}\right)}{a\Gamma\left(s+\frac{a+1}{a}\right)\Gamma(1)}m \sim \frac{m}{a}\Gamma\left(1+\frac{1}{a}\right)s^{-1-\frac{1}{a}},$$

i.e. all weights of subcliques have a power-law distribution.

Finally if we consider a special case when $N = 3$ and $M = 2$, we can obtain the same result as Theorem 2.3 in [2].

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