A THREE-STAGE ALGORITHM FOR SOLVING INCOMPRESSIBLE FLOW PROBLEMS

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ABSTRACT

The present study focuses on a reformulation of the Fractional-Step, Artificial Compressibility with Pressure-Projection (FSAC-PP) method by which a diffusion equation for the pressure is obtained through a velocity projection concept. The proposed method is a Fractional-Step Velocity-Projection (FSVP) approach, which is validated for a laminar flow in a lid driven cavity. The results are compared to the classical hyperbolic Artificial Compressibility (AC) method and reference data taken from the literature to demonstrate the strength of the proposed FSVP method.

1. INTRODUCTION

The development of incompressible flow solvers has stagnated in the last decades in which the usage of so-called pressure-correction algorithms has dominated. Patankar and Spalding [1] introduced a Semi-Implicit Method for Pressure Linked Equations, known as SIMPLE algorithm, which is the most representative of that group of incompressible flow solvers. Many derivatives of this algorithm were presented in the literature and are well-known nowadays. However, the classical methods of Chorin as the hyperbolic Artificial Compressibility (AC) formulation [2] and the semi-implicit Fractional-Step Pressure-Projection (FS-PP) [3] method have surprisingly received less attention, despite its mathematical simplicity and purity. In recent years, the development on incompressible flow solvers have resurged with the introduction of the Fractional-Step, Artificial Compressibility with Pressure-Projection (FSAC-PP) formulation proposed by Könözy [4], and Könözy and Drikakis [5], which unifies Chorin’s AC and FS-PP methods into one framework. Furthermore, the usage of a characteristic-based scheme and the inclusion of the the solution of the Riemann problem further extends this method within the class of Godunov-type methods which has shown superiority over the classical AC and FS-PP methods [4-6]. From a mathematical standpoint, it can be argued that the success of the FSAC-PP method is due to its alignment with the mathematical nature of the governing equations. For incompressible flows, it can be shown that the Navier–Stokes equations become mixed elliptic/hyperbolic partial differential equations, in contrast to its compressible form, in which the system of equations is fully hyperbolic. The AC method is fully hyperbolic while the pressure-Poisson solver in the FS-PP method is elliptic. The aforementioned FSAC-PP method unifies these
mathematical properties, and it is hyperbolic in the first Fractional-Step (FS) and elliptic in the second FS as required in the classification of the governing equations. However, the classification is based on the nature of transport equations themselves, and there is a lack of knowledge on the pressure transport itself, since its properties are not reflected in the classification of the governing equations. Therefore, a pressure diffusion equation has been derived in the present paper which makes desirable the investigation of the pressure properties in isolation. As a result of the present study, a three-stage algorithm has been proposed for solving incompressible flow problems through the inclusion of a pressure diffusion equation.

2. DERIVATION OF A PRESSURE DIFFUSION EQUATION

For solving incompressible flow problems, the derivation of a three-stage algorithm through a pressure diffusion equation can be formulated based on the FSAC-PP method. It is only logical to make an attempt to derive a three-stage algorithm if the governing equations can conform to an expected mathematical behaviour. Thus, the perturbed continuity equation of the AC method, which is proposed by Chorin [2], can be solved along with the momentum equation of the FS-PP approach [3] as it is consistent with the FSAC-PP formulation [4,5] in a semi-discrete form as

\[
\frac{1}{\beta} \frac{p^{(n+1)} - p^{(n)}}{\Delta \tau} + \nabla \cdot \mathbf{u}^{(n)} = 0, \tag{1}
\]

\[
\frac{\mathbf{u}^* - \mathbf{u}^{(n)}}{\Delta \tau} + \left[ (\bar{\mathbf{u}} \cdot \nabla) \mathbf{u}^{(n)} \right] = \nu \nabla^2 \mathbf{u}^{(n)}, \tag{2}
\]

where \(\beta\) is a convergence parameter, \(\Delta \tau\) is the pseudo-time step, \(p\) is the pressure field, \(\mathbf{u}\) is the velocity vector, and \(\nu\) is the kinematic viscosity of the fluid. The tilde denotes a characteristic-based Godunov-type numerical treatment of each velocity component. The superscript star indicates the prediction of an intermediate velocity field. The pressure gradient can be estimated in the next temporal fractional-step remaining consistent with both FS-PP [3] and FSAC-PP formulations [4] as

\[
\frac{\mathbf{u}^{(n+1)} - \mathbf{u}^*}{\Delta \tau} = - \frac{1}{\rho} \nabla p^{(n+1)}, \tag{3}
\]

and by taking the divergence of Eq. (3) yields

\[
\nabla^2 p^{(n+1)} = \frac{\rho}{\Delta \tau} \nabla \cdot \mathbf{u}^*, \tag{4}
\]

which is an elliptic-type pressure-Poisson equation. The divergence of the velocity field at pseudo-time level \((n+1)\) has to vanish due to the divergence-free constraint \(\nabla \cdot \mathbf{u}^{(n+1)}\) imposed by the FS-PP method [3]. Once the pressure is obtained by solving Eq. (4) numerically, the velocity field is updated relying on Eq. (3) as
\[ \mathbf{u}^{(n+1)} = \mathbf{u}^* - \frac{\Delta \tau}{\rho} \nabla p^{(n+1)}. \]  \quad (5)

The above described FSAC-PP numerical procedure \cite{4} can be repeated in each pseudo-time step until a convergence criterion is satisfied in the form of

\[ \max \left( \beta \| \nabla \cdot \mathbf{u}^{(n+1)} - \nabla \cdot \mathbf{u}^{(n)} \| \right) < \varepsilon, \]  \quad (6)

where the \( \varepsilon \) threshold is taken as a value of \( 10^{-8} \) in the present work. Relying on the perturbed continuity equation (1) and the velocity field update equation (5), a diffusion equation for the pressure can be derived by predicting the velocity field in the pseudo-time level \((n+1)\) through re-writing Eq. (1) as

\[ \frac{1}{\beta} \frac{p^{(n+1)} - p^{(n)}}{\Delta \tau} + \nabla \cdot \mathbf{u}^{(n+1)} = 0, \]  \quad (7)

which means that hereby a velocity-projection step can take place through the substitution of Eq. (5) into Eq. (7) leading to a pressure diffusion equation as

\[ \frac{1}{\beta} \frac{p^{(n+1)} - p^{(n)}}{\Delta \tau} = \frac{\Delta \tau}{\rho} \nabla^2 p^{(n+1)} - \nabla \cdot \mathbf{u}^*. \]  \quad (8)

The obtained pressure diffusion equation (8) is the canonical form of a parabolic partial differential equation which indicates that the pressure does indeed feature a different behaviour compared to the usual classification of the incompressible Navier–Stokes solvers. This fact may not be surprising because the classification procedure does not rely on the pressure field characterization, since usually only the behaviour of the velocity field is considered. It also means that the elliptic pressure-Poisson equation may work well for steady-state flows, but it could be an inadequate representation of time-dependent (unsteady) flows. In fact, no partial differential equation can be elliptic to model properly transient flows as the time derivative imposes natural bounds (or characteristics) which propagate through the physical domain. Therefore, a three-stage algorithm has been proposed in this paper for solving incompressible flow problems relying on Eqs. (2) and (5) through the above obtained pressure diffusion equation (8). The proposed three-stage numerical procedure for solving the incompressible Navier–Stokes equations is a Fractional-Step Velocity-Projection (FSVP) method, which can be summarised as

\[ \frac{\mathbf{u}^* - \mathbf{u}^{(n)}}{\Delta \tau} + \left( \mathbf{u} \cdot \nabla \right) \mathbf{u}^{(n)} = \nu \nabla^2 \mathbf{u}^{(n)}, \]  \quad (9)

\[ \frac{1}{\beta} \frac{p^{(n+1)} - p^{(n)}}{\Delta \tau} = \frac{\Delta \tau}{\rho} \nabla^2 p^{(n)} - \nabla \cdot \mathbf{u}^*, \]  \quad (10)

\[ \mathbf{u}^{(n+1)} = \mathbf{u}^* - \frac{\Delta \tau}{\rho} \nabla p^{(n+1)}. \]  \quad (11)
The set of equations (9), (10) and (11) are solved, where an intermediate velocity field \( \mathbf{u}^* \) is predicted in the first step similar to the FS-PP method. In the second step, a pressure diffusion equation is solved, where the pressure is taken at the pseudo-time level \( (n) \) on the right-hand side of Eq. (10) which allows for using an explicit numerical method. In the third step, the velocity field is updated relying on the intermediate velocity based on Eq. (9) and the solution of the pressure diffusion equation (10). The numerical convergence parameter \( \beta \) has been taken as unity for the favour of simplicity. In terms of pseudo-time marching, a fourth-order explicit Runge–Kutta scheme is used to integrate equations (9) and (10) of the proposed FSVP three-stage algorithm. A finite volume method (FVM) based discretization in conjunction with a third-order interpolation scheme is used to obtain the intercell fluxes at cell interfaces. In the present study, we employ a classical finite volume numerical treatment of the discretized equations, therefore, the solution of the Riemann problem which is an essential part of the Godunov-type convective flux treatment has not been considered. It is indicated by the removal of the tilde from the notation of the convective flux term in Eq. (9). In order to accelerate convergence, an over-relaxation approach is employed for both, AC and FSVP methods with the relaxation factor of \( \alpha = 2.0 \).

3. GEOMETRICAL AND PHYSICAL SETUP OF THE PROBLEM

The validation benchmark test case for the current work is a laminar flow in a lid driven cavity as depicted in Figure 1, along with its boundary conditions. An equidistant-spaced mesh distribution with 40 cells in each direction is chosen. The lid velocity is fixed at 1 m/s, together with a kinematic viscosity of 0.01 m²/s and reference length of 1 m, the Reynolds number is obtained as \( Re = 100 \). Thus, a laminar approximation is valid and no further considerations have to be given to any effect of turbulence. Reference numerical data of Ghia et al. [7] is available for these test conditions, which are widely used to validate incompressible flow solvers.

\[
\begin{align*}
\text{u} &= 1, \quad \text{v} = 0, \quad \frac{\partial \rho}{\partial n} = 0 \\
\text{u} &= v = 0, \quad \frac{\partial \rho}{\partial n} = 0 \\
\text{u} &= v = 0, \quad \frac{\partial \rho}{\partial n} = 0
\end{align*}
\]

Figure 1: Geometrical setup and boundary conditions for the lid driven cavity flow.
4. RESULTS AND DISCUSSION

We present results obtained with the AC and FSVP method in the present Section. The horizontal velocity component, superimposed by the streamlines, is shown in Figure 2 for both the AC (left) and FSVP (right) method. Both methods reproduce correctly the central vortex and also the secondary corner vortices. This is noteworthy as the rather coarse mesh considered in this investigation has a mesh spacing $\Delta x = \Delta y = \text{const.}$ which is of the same order of magnitude as the secondary, bottom left vortex structure itself. Hence, from a qualitative point of view, the FSVP method reproduces the same flow features as its benchmark solution obtained with the AC method. Figure 3 shows a more detailed insight into the velocity distribution by comparing the centreline velocity component along the horizontal and vertical plane. Close to the solid boundary, the agreement is excellent with the AC method. Towards the center of the cavity, the agreement is slightly off. It can be seen that the AC method reproduces the results of Ghia et al. [7] almost exactly.

![Figure 2: Contour plots of the horizontal velocity components and streamlines for the AC (left) and the proposed FSVP (right) method.](image1)

![Figure 3: Velocity components along the centreline of the horizontal and vertical plane compared to numerical data of Ghia et al. [7].](image2)
The FSVP method does not compare as favorably, however, the drop in accuracy is almost negligible. Figure 4, on the other hand, shows the history of the residuals of the $L_0$-norm which is defined in Eq. (6) and is an indicator for the satisfaction of the divergence-free constraint. It can be seen that there is little difference in the convergence up to 2000 iterations. However, the first iterations are non-conclusive as these are needed to remove the somewhat arbitrary and sometimes even non-physical initial conditions. Once the initial conditions have been marched towards a physical solution, the algorithm can work on improving the solution iteratively.

This process can be seen by the smooth and repeated wave-like nature of the residuals which start to develop at around 2000 iterations. It can be seen that the residual rate becomes favorable for the FSVP method and convergence is reached 9.3% earlier compared to the AC method. On the same note, however, it has to be said that the AC method converges two times faster in terms of total CPU time. This is due to the fact that the FSVP method solves three transport equations along with two velocity update equations while the AC method only requires the solution of three transport equations. Thus, the added overhead of two more equations comes at the cost of increased CPU time which is rewarded with a faster convergence.

These results present initial findings which are likely to improve by incorporating the solution of the Riemann problem, extending the FSVP method to a Godunov-type of method. This has been left for future work. In order for the pressure diffusion equation to have a physical meaning, the pseudo-time has to be converted into real-time. This can only be done if the convergence parameter $\beta$ is assigned with a physical meaningful and computable quantity.

5. CONCLUSIONS

In the preceding sections, a novel velocity projection idea has been incorporated into the well-established and validated FSAC-PP method which produced a parabolic pressure diffusion equation. This equation, together with the momentum
equation of the pressure projection method and its velocity update, form the framework of the FSVP method. Results have been presented for the laminar flow in a lid driven cavity at $Re = 100$. We showed that the convergence is achieved 9.3% faster compared to the AC method, while the actual CPU time was two times slower due to the added overhead of extra equations that need to be solved. In terms of accuracy, the FSVP method showed little differences compared to the AC method based on its centreline velocity components and it is likely that a Godunov-type treatment would further enhance the accuracy of the method.

REFERENCES


